ALBANIAN JOURNAL
OF MATHEMATICS
Volume 2, Number 3, Pages 199–213
ISSN 1930-1235: (2008)

DETERMINING EQUATIONS OF FAMILIES OF CYCLIC CURVES

R. Sanjeewa
Department of Mathematics and Statistics
Oakland University,
Rochester, MI, 48309.
rsanjeew@oakland.edu

T. Shaska
Department of Mathematics
University of Vlora,
Vlora, Albania
shaska@univlora.edu.al

Abstract. In previous work we determined automorphism groups of cyclic algebraic curves defined over fields of any odd characteristic. In this paper we determine parametric equations of families of curves for each automorphism group for such curves.

1. Introduction

Let $\mathcal{X}_g$ be an algebraic curve of genus $g \geq 2$ defined over a algebraically closed field of characteristic $p \neq 2$. If an automorphism group of a algebraic curve has normal cyclic subgroup such that the quotient space has genus zero, then such a curve is called a *cyclic curve*. We have studied automorphism groups of *cyclic curves* in [29], where we have listed all automorphism groups as well as ramification signatures of corresponding covers. In this paper we give a corresponding parametric equation for each family in [29].

In the second section we briefly introduce basic facts on cyclic curves and their automorphism. Let $G = \mathrm{Aut}(\mathcal{X}_g)$ automorphism group of given cyclic curve $\mathcal{X}_g$, the reduced automorphism group is $\bar{G} := \mathrm{Aut}(\mathcal{X}_g)/\langle w \rangle$, where $C_n = \langle w \rangle$ such that $g(\mathcal{X}^{C_n}) = 0$. This group $\bar{G}$ is embedded in $PGL_2(k)$ and therefore is isomorphic to one of C_m, D_m, A_4, S_4, A_5, *a semi direct product of elementary Abelian group with cyclic group*, $PSL(2, q)$, or $PGL(2, q)$. Then, $\bar{G}$ acts on a genus 0 field $k(x)$. We determine a rational function $\phi(x)$ that generates the fixed field $k(x)^{\bar{G}}$ in all cases cf. Lemma 1.

In section three, we determine the ramification signature σ of the cover $\Phi(x) : \mathcal{X}_g \to \mathbb{P}^1$ with monodromy group $G := \mathrm{Aut}(\mathcal{X}_g)$. Moduli spaces of covers Φ are

---

2000 *Mathematics Subject Classification.* Primary: 14Hxx, Secondary: 14H37, 14H10,
Key words and phrases. algebraic curves, Hurwitz spaces, equations.
Both authors were supported by a NATO grant, ICS. EAP. ASI No 982903.

Hurwitz spaces, which we denoted by $\mathcal{H}_\sigma$. There is a map $\Phi_\sigma : \mathcal{H}_\sigma \to \mathcal{M}_g$, where $\mathcal{M}_g$ is the moduli space of genus g algebraic curves. The image of this map is a subvariety of $\mathcal{M}_g$, which we denoted by $\mathcal{H}(G, \sigma)$. The dimension of $\mathcal{H}(G, \sigma)$ is determined. Hence, we have

$$\mathcal{X}_g \xrightarrow{C_n} \mathbb{P}^1 \xrightarrow{\bar{G}} \mathbb{P}^1$$

We list all possible automorphism groups, their signatures, and dimension of the loci $\mathcal{H}(G, \sigma)$.

In the last section, we determine the equations of families of curves for a given group. Using the rational function $\phi(x)$ we are able to determine parametric equation of each family $\mathcal{H}(G, \sigma)$. Since we know $\phi(x)$, we can find the branch points and then determine the equation of the curve from these branch points. We list corresponding equations of families of curves which we have listed in section three.

Throughout this paper we let $g \geq 2$ be a fixed integer, $\mathcal{X}$ a genus g cyclic curve, $G = \mathrm{Aut}(\mathcal{X})$ and $C_n \lhd G$ such that $g(\mathcal{X}^{C_n}) = 0$.

2. Preliminaries

Let $\mathcal{X}_g$ be genus $g \geq 2$ cyclic curve defined over an algebraically closed field k of characteristic $p \neq 2$. We take the equation of $\mathcal{X}_g$ to be $y^n = F(x)$, where $deg(F) = 2g + 2$. Let $K := k(x, y)$ be the function field of $\mathcal{X}_g$. Then K is a degree n extension field of $k(x)$ ramified exactly at $d = 2g + 2$ places $\alpha_1, ..., \alpha_d$ of $k(x)$.

Let $G = Aut(K/k)$. Since $k(x)$ is the only genus 0 subfield of degree n of K, then G fixes $k(x)$. Thus $Gal(K/k(x)) = \langle w \rangle$, with $w^n = 1$. Then the group $\bar{G} := G/\langle w \rangle$ is called *reduced automorphism group*. By the theorem of Dickson, $\bar{G}$ is isomorphic to one of the following: C_m, D_m, A_4, S_4, A_5, $PSL(2, q)$ and $PGL(2, q)$, or a semi direct product of elementary Abelian group with cyclic group, defined as

$$K_m := \langle \{\sigma_a, t | a \in \mathcal{U}_m\} \rangle, \ \ \textit{where } \mathcal{U}_m := \{a \in k | (a \prod_{j=0}^{\frac{p^t-1}{m}-1} (a^m - b_j)) = 0\}$$

and $t(x) = \xi^2 x$, $\quad \sigma_a(x) = x + a$, for each $a \in \mathcal{U}$, $b_j \in k^*$ and ξ is a primitive $2m$-th root of unity; see [10]. $\mathcal{U}_m$ is a subgroup of the additive group of k.

The group $\bar{G}$ acts on $k(x)$ via the natural way. The fixed field is a genus 0 field, say $k(z)$. Thus z is a degree $|\bar{G}|$ rational function in x, say $z = \phi(x)$. The following lemma determines rational functions for all $\bar{G}$; see [29].

Let $\phi_0 : \mathcal{X}_g \to \mathbb{P}^1$ and $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be covers which correspond to the extensions $K/k(x)$ and $k(x)/k$ respectively. Then, $\psi := \phi \circ \phi_0$ has monodromy group $G := Aut(\mathcal{X}_g)$. By basic covering theory, the group G is embedded in the group S_l, where $l = deg(\psi)$. There is an r-tuple $\overline{\sigma} := (\sigma_1, ..., \sigma_r)$, where $\sigma_i \in S_l$ such that $\sigma_1, ..., \sigma_r$ generate G and $\sigma_1 ... \sigma_r = 1$. The signature of Φ is an r-tuple of conjugacy classes $\mathcal{C} := (C_1, ..., C_r)$ in S_l such that C_i is the conjugacy class of σ_i. We can find the signature of $\psi_0 : \mathcal{X}_g \to \mathbb{P}^1$ by using the signature of $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ and Riemann-Hurwitz formula.

Moduli spaces of covers ψ are Hurwitz space, which we denoted by $\mathcal{H}_\sigma$. There is a map $\Phi_\sigma : \mathcal{H}_\sigma \to \mathcal{M}_g$, where $\mathcal{M}_g$ is the moduli space of genus g algebraic curves. The image of this map is a subvariety of $\mathcal{M}_g$, which we denoted by $\mathcal{H}(G, \sigma)$. Using the signature of ψ and Riemann-Hurwitz formula, one can find out dimension of $\mathcal{H}(G, \sigma)$, which we denoted by δ.

We summarize all in the following Lemma:

Lemma 1. *Let k be an algebraically closed field of characteristic p, H_t a subgroup of the additive group of k with $|H_t| = p^t$ and $b_j \in k^*$, and $\bar{G}$ be a finite subgroup of $PGL_2(k)$ acting on the field $k(x)$. Then, $\bar{G}$ is isomorphic to one of the following groups C_m, D_m, A_4, S_4, A_5, $U = C_p^t$, K_m, $PSL_2(q)$ and $PGL_2(q)$, where $q = p^f$ and $(m,p) = 1$. Moreover, the fixed subfield $k(x)^{\bar{G}} = k(z)$ is given by Table 1, where $\alpha = \frac{q(q-1)}{2}$, $\beta = \frac{q+1}{2}$.*

Case	$\bar{G}$	z	*Ramification*
1	C_m, $(m,p) = 1$	x^m	(m,m)
2	D_{2m}, $(m,p) = 1$	$x^m + \frac{1}{x^m}$	$(2,2,m)$
3	A_4, $p \neq 2,3$	$\frac{x^{12}-33x^8-33x^4+1}{x^2(x^4-1)^2}$	$(2,3,3)$
4	S_4, $p \neq 2,3$	$\frac{(x^8+14x^4+1)^3}{108(x(x^4-1))^4}$	$(2,3,4)$
5	A_5, $p \neq 2,3,5$	$\frac{(-x^{20}+228x^{15}-494x^{10}-228x^5-1)^3}{(x(x^{10}+11x^5-1))^5}$	$(2,3,5)$
	A_5, $p = 3$	$\frac{(x^{10}-1)^6}{(x(x^{10}+2ix^5+1))^5}$	$(6,5)$
6	U	$\prod_{a \in H_t} (x+a)$	(p^t)
7	K_m	$(x \prod_{j=0}^{\frac{p^t-1}{m}-1} (x^m - b_j))^m$	(mp^t, m)
8	$PSL(2,q)$, $p \neq 2$	$\frac{((x^q-x)^{q-1}+1)^{\frac{q+1}{2}}}{(x^q-x)^{\frac{q(q-1)}{2}}}$	(α, β)
9	$PGL(2,q)$	$\frac{((x^q-x)^{q-1}+1)^{q+1}}{(x^q-x)^{q(q-1)}}$	$(2\alpha, 2\beta)$

TABLE 1. Rational functions correspond to each $\bar{G}$

3. Automorphism groups and their signatures of cyclic curves

As above $\bar{G} := G/G_0$, where $G_0 := Gal(k(x,y)/k(x))$. The following theorem determines ramification signatures and dimensions of δ of $\mathcal{H}(G,\sigma)$ for all $\bar{G}$ when $p > 5$; see [29] for details.

Theorem 3.1. *The signature of cover $\Phi(x) : \mathcal{X} \to \mathcal{X}^G$ and dimension δ is given in Table 2. In Table 2, $m = |PSL_2(q)|$ for cases 38-41 and $m = |PGL_2(q)|$ for cases 42-45.*

#	G	$\delta(G,C)$	δ, n, g	$\mathcal{C} = (C_1, \dots, C_r)$
1	$(p,m)=1$	$\frac{2(g+n-1)}{m(n-1)} - 1$	$n < g+1$	$(m, m, n, \dots, n)$
2	C_m	$\frac{2g+n-1}{m(n-1)} - 1$		$(m, mn, n, \dots, n)$
3		$\frac{2g}{m(n-1)} - 1$	$n < g$	$(mn, mn, n, \dots, n)$
4	$(p,m)=1$	$\frac{g+n-1}{m(n-1)}$		$(2, 2, m, n, \dots, n)$
5		$\frac{2g+m+2n-nm-2}{2m(n-1)}$		$(2n, 2, m, n, \dots, n)$
6	D_{2m}	$\frac{g}{m(n-1)}$		$(2, 2, mn, n, \dots, n)$
7		$\frac{g+m+n-mn-1}{m(n-1)}$	$n < g+1$	$(2n, 2n, m, n, \dots, n)$
8		$\frac{2g+m-mn}{2m(n-1)}$	$g \neq 2$	$(2n, 2, mn, n, \dots, n)$
9		$\frac{g+m-mn}{m(n-1)}$	$n < g$	$(2n, 2n, mn, n, \dots, n)$
10		$\frac{n+g-1}{6(n-1)}$		$(2, 3, 3, n, \dots, n)$
11	A_4	$\frac{g-n+1}{6(n-1)}$		$(2, 3n, 3, n, \dots, n)$
12		$\frac{g-3n+3}{6(n-1)}$		$(2, 3n, 3n, n, \dots, n)$
13		$\frac{g-2n+2}{6(n-1)}$	$\delta \neq 0$	$(2n, 3, 3, n, \dots, n)$
14		$\frac{g-4n+4}{6(n-1)}$		$(2n, 3n, 3, n, \dots, n)$
15		$\frac{g-6n+6}{6(n-1)}$	$\delta \neq 0$	$(2n, 3n, 3n, n, \dots, n)$
16		$\frac{g+n-1}{12(n-1)}$		$(2, 3, 4, n, \dots, n)$
17		$\frac{g-3n+3}{12(n-1)}$		$(2, 3n, 4, n, \dots, n)$
18		$\frac{g-2n+2}{12(n-1)}$		$(2, 3, 4n, n, \dots, n)$
19		$\frac{g-6n+6}{12(n-1)}$		$(2, 3n, 4n, n, \dots, n)$
20	S_4	$\frac{g-5n+5}{12(n-1)}$		$(2n, 3, 4, n, \dots, n)$
21		$\frac{g-9n+9}{12(n-1)}$		$(2n, 3n, 4, n, \dots, n)$
22		$\frac{g-8n+8}{12(n-1)}$		$(2n, 3, 4n, n, \dots, n)$
23		$\frac{g-12n+12}{12(n-1)}$		$(2n, 3n, 4n, n, \dots, n)$
24		$\frac{g+n-1}{30(n-1)}$		$(2, 3, 5, n, \dots, n)$
25		$\frac{g-5n+5}{30(n-1)}$		$(2, 3, 5n, n, \dots, n)$
26		$\frac{g-15n+15}{30(n-1)}$		$(2, 3n, 5n, n, \dots, n)$
27		$\frac{g-9n+9}{30(n-1)}$		$(2, 3n, 5, n, \dots, n)$
28	A_5	$\frac{g-14n+14}{30(n-1)}$		$(2n, 3, 5, n, \dots, n)$
29		$\frac{g-20n+20}{30(n-1)}$		$(2n, 3, 5n, n, \dots, n)$
30		$\frac{g-24n+24}{30(n-1)}$		$(2n, 3n, 5, n, \dots, n)$
31		$\frac{g-30n+30}{30(n-1)}$		$(2n, 3n, 5n, n, \dots, n)$
32		$\frac{2g+2n-2}{p^t(n-1)} - 2$		$(p^t, n, \dots, n)$
33	U	$\frac{2g+np^t-p^t}{p^t(n-1)} - 2$	$(n,p)=1, n \mid p^t-1$	$(np^t, n, \dots, n)$
34		$\frac{2(g+n-1)}{mp^t(n-1)} - 1$	$(m,p)=1, m \mid p^t-1$	$(mp^t, m, n, \dots, n)$
35		$\frac{2g+2n+p^t-np^t-2}{mp^t(n-1)} - 1$	$(m,p)=1, m \mid p^t-1$	$(mp^t, nm, n, \dots, n)$
36	K_m	$\frac{2g+np^t-p^t}{mp^t(n-1)} - 1$	$(nm,p)=1, nm \mid p^t-1$	$(nmp^t, m, n, \dots, n)$
37		$\frac{2g}{mp^t(n-1)} - 1$	$(nm,p)=1, nm \mid p^t-1$	$(nmp^t, nm, n, \dots, n)$
38		$\frac{2(g+n-1)}{m(n-1)} - 1$	$\left(\frac{q-1}{2}, p\right) = 1$	$(\alpha, \beta, n, \dots, n)$
39	$PSL_2(q)$	$\frac{2g+q(q-1)-n(q+1)(q-2)-2}{m(n-1)} - 1$	$\left(\frac{q-1}{2}, p\right) = 1$	$(\alpha, n\beta, n, \dots, n)$
40		$\frac{2g+nq(q-1)+q-q^2}{m(n-1)} - 1$	$\left(\frac{n(q-1)}{2}, p\right) = 1$	$(n\alpha, \beta, n, \dots, n)$
41		$\frac{2g}{m(n-1)} - 1$	$\left(\frac{n(q-1)}{2}, p\right) = 1$	$(n\alpha, n\beta, n, \dots, n)$
42		$\frac{2(g+n-1)}{m(n-1)} - 1$	$(q-1, p) = 1$	$(2\alpha, 2\beta, n, \dots, n)$
43	$PGL_2(q)$	$\frac{2g+q(q-1)-n(q+1)(q-2)-2}{m(n-1)} - 1$	$(q-1, p) = 1$	$(2\alpha, 2n\beta, n, \dots, n)$
44		$\frac{2g+nq(q-1)+q-q^2}{m(n-1)} - 1$	$(n(p-1), p) = 1$	$(2n\alpha, 2\beta, n, \dots, n)$
45		$\frac{2g}{m(n-1)} - 1$	$(n(q-1), p) = 1$	$(2n\alpha, 2n\beta, n, \dots, n)$

TABLE 2. The signature of curves and dimensions δ for char > 5

Remark 1. *The above theorem gives signatures and dimensions for* $p > 5$. *We know that* $\bar{G} \cong C_m, D_m, A_4, S_4, U, K_m, PSL(2, q), PGL(2, q)$ *when* $p = 5$ *and* $\bar{G} \cong C_m, D_m, A_5, U, K_m, PSL(2, q), PGL(2, q)$ *when* $p = 3$; *see [10]. All cases except* $\bar{G} \cong A_5$ *have ramification as* $p > 5$. *Hence signatures and dimensions are the same as* $p > 5$. *However,* $\bar{G} \cong A_5$ *has different ramification. Hence, that case has signatures and dimensions as in Table 3.*

Case	$\bar{G}$	$\delta(G, C)$	$\mathcal{C} = (C_1, ..., C_r)$
a		$\frac{g+n-1}{30(n-1)} - 1$	$(6, 5, n, ..., n)$
b		$\frac{g+5n-5}{30(n-1)} - 1$	$(6, 5n, n, ..., n)$
c	A_5	$\frac{g+6n-6}{30(n-1)} - 1$	$(6n, 5, n, ..., n)$
d		$\frac{g}{30(n-1)} - 1$	$(6n, 5n, n, ..., n)$

TABLE 3. The signature of curve and dimension δ for $\bar{G} \cong A_5$, $p = 3$

The following theorem determines the list of all automorphism groups of cyclic algebraic curves defined over any algebraically closed field of characteristic $p \neq 2$, details will be provided in [29].

Theorem 3.2. *Let* $\mathcal{X}_g$ *be a genus* $g \geq 2$ *irreducible cyclic curve defined over an algebraically closed field* k *of characteristic char* $(k) = p$, $G = Aut(\mathcal{X}_g)$, *and* $\bar{G}$ *its reduced automorphism group. If* $|G| > 1$ *then is* G *is one of the following:*

(1) $\bar{G} \cong C_m$: *Then,* $G \cong C_{mn}$ *or* $\langle r, s | \, r^n = 1, s^m = 1, srs^{-1} = r^l \rangle$, $(l, n) = 1$ *and* $l^m \equiv 1$ *(mod n).*

(2) *If* $\bar{G} \cong D_{2m}$ *then* $G \cong D_{2m} \times C_n$ *or*

$$G'_4 = \langle r, s, t | \, r^n = 1, s^2 = 1, t^2 = 1, (st)^m = 1, srs^{-1} = r^l, trt^{-1} = r^l \rangle$$

$$G'_7 = \langle r, s, t | \, r^n = 1, s^2 = r^{\frac{n}{2}}, t^2 = r^{\frac{n}{2}}, (st)^m = 1, srs^{-1} = r^l, trt^{-1} = r^l \rangle$$

where $(l, n) = 1$ *and* $l^2 \equiv 1$ *(mod n) or*

$$G_4 = \langle r, s, t | \, r^n = 1, s^2 = 1, t^2 = 1, (st)^m = 1, srs^{-1} = r^l, trt^{-1} = r^k \rangle$$

$$G_5 = \langle r, s, t | \, r^n = 1, s^2 = r^{\frac{n}{2}}, t^2 = 1, (st)^m = 1, srs^{-1} = r^l, trt^{-1} = r^k \rangle$$

$$G_6 = \langle r, s, t | \, r^n = 1, s^2 = 1, t^2 = 1, (st)^m = r^{\frac{n}{2}}, srs^{-1} = r^l, trt^{-1} = r^k \rangle$$

$$G_7 = \langle r, s, t | \, r^n = 1, s^2 = r^{\frac{n}{2}}, t^2 = r^{\frac{n}{2}}, (st)^m = 1, srs^{-1} = r^l, trt^{-1} = r^k \rangle$$

$$G_8 = \langle r, s, t | \, r^n = 1, s^2 = r^{\frac{n}{2}}, t^2 = 1, (st)^m = r^{\frac{n}{2}}, srs^{-1} = r^l, trt^{-1} = r^k \rangle$$

$$G_9 = \langle r, s, t | \, r^n = 1, s^2 = r^{\frac{n}{2}}, t^2 = r^{\frac{n}{2}}, (st)^m = r^{\frac{n}{2}}, srs^{-1} = r^l, trt^{-1} = r^k \rangle$$

where $(l, n) = 1$ *and* $l^2 \equiv 1$ *(mod n),* $(k, n) = 1$ *and* $k^2 \equiv 1$ *(mod n).*

(3) *If* $\bar{G} \cong A_4$ *and* $p \neq 2, 3$ *then* $G \cong A_4 \times C_n$ *or*

$$G'_{10} = \langle r, s, t | \, r^n = 1, s^2 = 1, t^3 = 1, (st)^3 = 1, srs^{-1} = r, trt^{-1} = r^l \rangle$$

$$G'_{12} = \langle r, s, t | \, r^n = 1, s^2 = 1, t^3 = r^{\frac{n}{3}}, (st)^3 = r^{\frac{n}{3}}, srs^{-1} = r, trt^{-1} = r^l \rangle$$

where $(l, n) = 1$ *and* $l^3 \equiv 1$ *(mod n) or*

$$\langle r,s,t|\, r^n=1, s^2=r^{\frac{n}{2}}, t^3=r^{\frac{n}{2}}, (st)^5=r^{\frac{n}{2}}, srs^{-1}=r, trt^{-1}=r\rangle,\ or$$

$$G_{10}=\langle r,s,t|\, r^n=1, s^2=1, t^3=1, (st)^3=1, srs^{-1}=r, trt^{-1}=r^k\rangle$$
$$G_{13}=\langle r,s,t|\, r^n=1, s^2=r^{\frac{n}{2}}, t^3=1, (st)^3=1, srs^{-1}=r, trt^{-1}=r^k\rangle$$

where $(k,n)=1$ *and* $k^3\equiv 1$ *(mod n).*

(4) *If* $\bar{G}\cong S_4$ *and* $p\neq 2,3$ *then* $G\cong S_4\times C_n$ *or*

$$G_{16}=\langle r,s,t|\, r^n=1, s^2=1, t^3=1, (st)^4=1, srs^{-1}=r^l, trt^{-1}=r\rangle$$
$$G_{18}=\langle r,s,t|\, r^n=1, s^2=1, t^3=1, (st)^4=r^{\frac{n}{2}}, srs^{-1}=r^l, trt^{-1}=r\rangle$$
$$G_{20}=\langle r,s,t|\, r^n=1, s^2=r^{\frac{n}{2}}, t^3=1, (st)^4=1, srs^{-1}=r^l, trt^{-1}=r\rangle$$
$$G_{22}=\langle r,s,t|\, r^n=1, s^2=r^{\frac{n}{2}}, t^3=1, (st)^4=r^{\frac{n}{2}}, srs^{-1}=r^l, trt^{-1}=r\rangle$$

where $(l,n)=1$ *and* $l^2\equiv 1$ *(mod n).*

(5) *If* $\bar{G}\cong A_5$ *and* $p\neq 2,5$ *then* $G\cong A_5\times C_n$ *or*

$$\langle r,s,t|\, r^n=1, s^2=r^{\frac{n}{2}}, t^3=r^{\frac{n}{2}}, (st)^5=r^{\frac{n}{2}}, srs^{-1}=r, trt^{-1}=r\rangle$$

(6) *If* $\bar{G}\cong U$ *then* $G\cong U\times C_n$ *or*

$$\left\langle r,s_1,s_2,...,s_t|r^n=s_1^p=s_2^p=...=s_t^p=1, s_is_j=s_js_i, s_irs_i^{-1}=r^l, 1\leq i,j\leq t\right\rangle$$

where $(l,n)=1$ *and* $l^p\equiv 1$ *(mod n).*

(7) *If* $\bar{G}\cong K_m$ *then* $G\cong <r,s_1,...,s_t,v|r^n=s_1^p=...=s_t^p=v^m=1, s_is_j=s_js_i, vrv^{-1}=r, s_irs_i^{-1}=r^l, s_ivs_i^{-1}=v^k, 1\leq i,j\leq t>$ *where* $(l,n)=1$ *and* $l^p\equiv 1$ *(mod n),* $(k,m)=1$ *and* $k^p\equiv 1$ *(mod m) or*

$$G_{35}=\left\langle r,s_1,...,s_t|r^{nm}=s_1^p=...=s_t^p=1, s_is_j=s_js_i, s_irs_i^{-1}=r^l, 1\leq i,j\leq t\right\rangle$$

where $(l,nm)=1$ *and* $l^p\equiv 1(\mod nm)$.

(8) *If* $\bar{G}\cong PSL_2(q)$ *then* $G\cong PSL_2(q)\times C_n$ *or* $SL_2(3)$.

(9) *If* $\bar{G}\cong PGL(2,q)$ *then* $G\cong PGL(2,q)\times C_n$.

Proof. See [29]. □

4. Equations of curves

The group $\bar{G}$ is the monodromy group of the cover $\phi:\mathbb{P}^1\to\mathbb{P}^1$ with signature $(\sigma_1,\sigma_2,\sigma_3)$ as in section 2. We fix coordinates in $\mathbb{P}^1$ as x and z respectively and from now on we denote the cover $\phi:\mathbb{P}^1_x\to\mathbb{P}^1_z$. Thus, z is a rational function in x of the degree $|\bar{G}|$. We denote by q_1,q_2,q_3 corresponding branch points of ϕ. Let S be the set of branch points of $\Phi:\mathcal{X}_g\to\mathbb{P}^1_z$. Clearly $q_1,q_2,q_3\in S$. Let $y^n=f(x)$ be the equation of $\mathcal{X}_g$ and W be the images in $\mathbb{P}^1_x$ of roots of $f(x)$ and

$$V:=\bigcup_{i=1}^{3}\phi^{-1}(q_i).$$

Let

$$z = \frac{\Psi(x)}{\Upsilon(x)}, \text{ where } \Psi(x), \Upsilon(x) \in k[x].$$

Then we have

$$z - q_i = \frac{\Gamma(x)}{\Upsilon(x)}$$

for each branch point q_i, $i = 1, 2, 3$, where $\Gamma(x) \in k[x]$. Hence,

$$\Gamma(x) = \Psi(x) - q_i \cdot \Upsilon(x)$$

is degree $|\bar{G}|$ equation and multiplicity of all roots of $\Gamma(x)$ correspond to the ramification index for each q_i. Now we define the following three functions:

$$\begin{aligned} \varphi^r(x) &:= \Psi(x) - q_1 \cdot \Upsilon(x) \\ \chi^s(x) &:= \Psi(x) - q_2 \cdot \Upsilon(x) \\ \psi^t(x) &:= \Psi(x) - q_3 \cdot \Upsilon(x) \end{aligned} \tag{1}$$

where superscript denote the ramification index of q_i. Clearly, $\phi^{-1}(S \backslash \{q_1, q_2, q_3\}) \subset W$. Let $\lambda \in \mathrm{S} \setminus \{q_1, q_2, q_2\}$. The points in the fiber $\phi^{-1}(\lambda)$ are the roots of the equation:

$$\Psi(x) - \lambda \cdot \Upsilon(x) = 0 \tag{2}$$

Let

$$G(x) := \prod_{\lambda \in S \backslash \{q_1, q_2, q_2\}} (\Psi(x) - \lambda \cdot \Upsilon(x)) \tag{3}$$

There are following cases and corresponding equations of the curve $y^n = f(x)$ for each fixed ϕ.

	Intersection	$f(x)$
1)	$V \cap W = \emptyset$	$G(x)$
2)	$V \cap W = \phi^{-1}(q_1)$	$\varphi(x) \cdot G(x)$
3)	$V \cap W = \phi^{-1}(q_2)$	$\chi(x) \cdot G(x)$
4)	$V \cap W = \phi^{-1}(q_3)$	$\psi(x) \cdot G(x)$
5)	$V \cap W = \phi^{-1}(q_1) \cup \phi^{-1}(q_2)$	$\varphi(x) \cdot \chi(x) \cdot G(x)$
6)	$V \cap W = \phi^{-1}(q_2) \cup \phi^{-1}(q_3)$	$\chi(x) \cdot \psi(x) \cdot G(x)$
7)	$V \cap W = \phi^{-1}(q_1) \cup \phi^{-1}(q_3)$	$\varphi(x) \cdot \psi(x) \cdot G(x)$
8)	$V \cap W = \phi^{-1}(q_1) \cup \phi^{-1}(q_2) \cup \phi^{-1}(q_3)$	$\varphi(x) \cdot \chi(x) \cdot \psi(x) \cdot G(x)$

The following theorem gives us equations of families of curves for automorphism groups which are related to Theorem 3.1 and Theorem 3.2.

Theorem 4.1. *Let $\mathcal{X}_g$ be a genus $g \geq 2$ cyclic curve with $Aut(\mathcal{X}_g) = G$, where G is related to the cases 1-45 in Table 2. Then $\mathcal{X}_g$ has an equation as cases 1-45 in Table 4.*

Proof: We consider all cases one by one for the reduced automorphism group $\bar{G}$.

#	$\bar{G}$	$y^n = f(x)$
1		$x^{m\delta} + a_1 x^{m(\delta-1)} + \cdots + a_\delta x^m + 1$
2	C_m	$x^{m\delta} + a_1 x^{m(\delta-1)} + \cdots + a_\delta x^m + 1$
3		$x(x^{m\delta} + a_1 x^{m(\delta-1)} + \cdots + a_\delta x^m + 1)$
4		$F(x) := \prod_{i=1}^{\delta}(x^{2m} + \lambda_i x^m + 1)$
5		$(x^m - 1) \cdot F(x)$
6		$x \cdot F(x)$
7	D_{2m}	$(x^{2m} - 1) \cdot F(x)$
8		$x(x^m - 1) \cdot F(x)$
9		$x(x^{2m} - 1) \cdot F(x)$
10		$G(x) := \prod_{i=1}^{\delta}(x^{12} - \lambda_i x^{10} - 33x^8 + 2\lambda_i x^6 - 33x^4 - \lambda_i x^2 + 1)$
11		$(x^4 + 2i\sqrt{3}x^2 + 1) \cdot G(x)$
12	A_4	$(x^8 + 14x^4 + 1) \cdot G(x)$
13		$x(x^4 - 1) \cdot G(x)$
14		$x(x^4 - 1)(x^4 + 2i\sqrt{3}x^2 + 1) \cdot G(x)$
15		$x(x^4 - 1)(x^8 + 14x^4 + 1) \cdot G(x)$
16		$M(x)$
17		$S(x) \cdot M(x)$
18		$T(x) \cdot M(x)$
19		$S(x) \cdot T(x) \cdot M(x)$
20	S_4	$R(x) \cdot M(x)$
21		$R(x) \cdot S(x) \cdot M(x)$
22		$R(x) \cdot T(x) \cdot M(x)$
23		$R(x) \cdot S(x) \cdot T(x).M(x)$
24		$\Lambda(x)$
25		$(x(x^{10} + 11x^5 - 1)) \cdot \Lambda(x)$
26		$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1)(x(x^{10} + 11x^5 - 1)) \cdot \Lambda(x)$
27		$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1) \cdot \Lambda(x)$
28	A_5	$Q(x) \cdot \Lambda(x)$
29		$x(x^{10} + 11x^5 - 1).\psi(x) \cdot \Lambda(x)$
30		$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1) \cdot \psi(x) \cdot \Lambda(x)$
31		$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1)(x(x^{10} + 11x^5 - 1)) \cdot \psi(x) \cdot \Lambda(x)$
32	U	$B(x)$
33		$B(x)$
34		$\Theta(x)$
35	K_m	$x \prod_{j=1}^{\frac{p^t-1}{m}} (x^m - b_j) \cdot \Theta(x)$
36		$\Theta(x)$
37		$x \prod_{j=1}^{\frac{p^t-1}{m}} (x^m - b_j) \cdot \Theta(x)$
38		$\Delta(x)$
39	$PSL_2(q)$	$((x^q - x)^{q-1} + 1) \cdot \Delta(x)$
40		$(x^q - x) \cdot \Delta(x)$
41		$(x^q - x)((x^q - x)^{q-1} + 1) \cdot \Delta(x)$
42		$\Omega(x)$
43	$PGL_2(q)$	$((x^q - x)^{q-1} + 1) \cdot \Omega(x)$
44		$(x^q - x) \cdot \Omega(x)$
45		$(x^q - x)((x^q - x)^{q-1} + 1) \cdot \Omega(x)$

Table 4. The equations of the curves related to the cases in Table 2

4.1. $\bar{G} \cong C_m$. Then, $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ has signature (m, m). We identify the branch points of ϕ are 0 and ∞. Let $q_1 = \infty$, $q_2 = 0$. By Lemma 1, we know that $\phi(x) = x^m$. Hence $\varphi(x) = 1$ and $\chi(x) = x$. Let $\lambda_i \in \mathrm{S} \setminus \{0, \infty\}$. The points in the fiber $\phi^{-1}(\lambda_i)$ are the roots of the polynomial

$$G_{\lambda_i}(x) := x^m - \lambda_i$$

Now we can compute equations for the cases 1-3 in Table 4. If $W \cap V = \varnothing$ then the equation of the curve is $y^n = G(x)$ where

$$G(x) = \prod_{i=1}^{\delta} G_{\lambda_i}(x)$$

and δ is as case 1 in Table 2. Let $a_1, ..., a_\delta$ denote the symmetric polynomials in $\lambda_1, ..., \lambda_\delta$. Further we can take $\lambda_1 ... \lambda_\delta = 1$. Hence the equation of the curve is

$$y^n = x^{m\delta} + a_1 x^{m(\delta-1)} + ... + a_\delta x^m + 1$$

If $V \cap W = \phi^{-1}(q_1)$(i.e. case 2 in Table 4) then we know that the equation is $y^n = \varphi(x).G(x)$. Hence the equation is

$$y^n = x^{m\delta} + a_1 x^{m(\delta-1)} + ... + a_\delta x^m + 1$$

where δ is as case 2 in Table 2. If $V \cap W = \phi^{-1}(q_1) \cup \phi^{-1}(q_2)$ (i.e. case 3 in Table 4) then the equation is $y^n = \varphi(x).\chi(x).G(x)$. Hence

$$y^n = x(x^{m\delta} + a_1 x^{m(\delta-1)} + ... + a_\delta x^m + 1)$$

where δ is as case 3 in Table 2.

4.2. $\bar{G} \cong D_{2m}$. Then, $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ has signature $(2, 2, m)$. The branch points of $\phi(x)$ are ∞ and ± 2. Let $q_1 = \infty$, $q_2 = 2$ and $q_3 = -2$. By Lemma 1, we know that

$$\phi(x) = x^m + \frac{1}{x^m}.$$

Since $\phi(x) - 2 = \frac{(x^m-1)^2}{x^m}$ and $\phi(x) + 2 = \frac{(x^m+1)^2}{x^m}$, $\varphi(x) = x$, $\chi(x) = x^m - 1$ and $\psi = x^m + 1$. In this case we have $G(x)$ as below.

$$G(x) = \prod_{i=1}^{\delta} (x^{2m} - \lambda_i x^m + 1)$$

where $\lambda_i \in \mathrm{S} \setminus \{0, \pm 2\}$ and δ is as corresponding case in Table 2. Then each family is parameterized as cases 4-9 in Table 4.

4.3. $\bar{G} \cong A_4$. Then, $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ has signature $(2, 3, 3)$. We choose branch points $q_1 = \infty$, $q_2 = 6i\sqrt{3}$, and $q_3 = -6i\sqrt{3}$, where $i^2 = -1$. We know that

$$\phi(x) = \frac{x^{12} - 33x^8 - 33x^4 + 1}{x^2(x^4 - 1)^2}.$$

Thus the points in the fiber of q_1, q_2, q_3 are the roots of the polynomials:

$$\begin{aligned}
\varphi(x) &= x(x^4 - 1) \\
\chi(x) &= x^4 - 2i\sqrt{3}x^2 + 1 \\
\psi(x) &= x^4 + 2i\sqrt{3}x^2 + 1
\end{aligned}$$

Let $\lambda_i \in \mathrm{S} \setminus \{\infty, \pm 6i\sqrt{3}\}$ then points of $\phi^{-1}(\lambda_i)$ are roots of the polynomial

$$G_{\lambda_i}(x) = x^{12} - \lambda_i x^{10} - 33x^8 + 2\lambda_i x^6 - 33x^4 - \lambda_i x^2 + 1$$

There are δ points in $\mathrm{S} \setminus \{\infty, \pm 6i\sqrt{3}\}$. Hence, we have

$$G(x) = \prod_{i=1}^{\delta} (x^{12} - \lambda_i x^{10} - 33x^8 + 2\lambda_i x^6 - 33x^4 - \lambda_i x^2 + 1)$$

Then, each family is parameterized as cases 10-15 in Table 4, where δ is as corresponding case in Table 2.

4.4. $\bar{G} \cong S_4$**.** Then, $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ has signature $(2, 3, 4)$. The branch points of $\phi(x)$ are $\{0, 1, \infty\}$. Let $q_1 = 1$, $q_2 = 0$ and $q_3 = \infty$. Then

$$\begin{aligned} \varphi(x) &= x^{12} - 33x^8 - 33x^4 + 1 \\ \chi(x) &= x^8 + 14x^4 + 1 \\ \psi(x) &= x(x^4 - 1) \end{aligned}$$

For $\lambda_i \in \mathrm{S} \setminus \{0, 1, \infty\}$, the points in $\phi^{-1}(\lambda_i)$ are roots of the polynomial

$$\begin{aligned} G_{\lambda_i}(x) =& x^{24} + \lambda_i x^{20} + (759 - 4\lambda_i)x^{16} + 2(3\lambda_i + 1228)x^{12} \\ &+ (759 - 4\lambda_i)x^8 + \lambda_i x^4 + 1 \end{aligned}$$

There are δ points in $S \backslash \{0, 1, \infty\}$, where δ is given as in Table 2. We denote

$$M(x) := \prod_{i=1}^{\delta} G_{\lambda_i}(x)$$

Then, each family is parameterized as cases 16-23, where $R(x), S(x), T(x)$ are $\varphi(x), \chi(x), \psi(x)$ respectively.

4.5. $\bar{G} \cong A_5$**.** The branch points of $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ are 0, 1728 and ∞. Let $q_1 = 0$, $q_2 = \infty$ and $q_3 = 1728$. At the place $q_3 = 1728$ the function has the following ramification

$$\phi(x) - 1728 = -\frac{(x^{30} + 522x^{25} - 10005x^{20} - 10005x^{10} - 522x^5 + 1)^2}{x^5(x^{10} + 11x^5 - 1)^5}$$

Then,

$$\begin{aligned} \varphi(x) &= x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1 \\ \chi(x) &= x(x^{10} + 11x^5 - 1) \\ \psi(x) &= x^{30} + 522x^{25} - 10005x^{20} - 10005x^{10} - 522x^5 + 1 \end{aligned}$$

For each $\lambda_i \in \mathrm{S} \setminus \{0, 1728, \infty\}$ the places in $\phi^{-1}(\lambda_i)$ are the roots of the following polynomial

$$\begin{aligned} G_{\lambda_i}(x) = &- x^{60} + (684 - \lambda_i)x^{55} - (55\lambda_i + 157434)x^{50} - (1205\lambda_i - 12527460)x^{45} \\ &- (13090\lambda_i + 77460495)x^{40} + (130689144 - 69585\lambda_i)x^{35} \\ &+ (33211924 - 134761\lambda_i)x^{30} + (69585\lambda_i - 130689144)x^{25} \\ &- (13090\lambda_i + 77460495)x^{20} - (12527460 - 1205\lambda_i)x^{15} \\ &- (157434 + 55\lambda_i)x^{10} + (\lambda_i - 684)x^5 - 1 \end{aligned}$$

Then,

$$\Lambda(x) = \prod_{i=1}^{\delta} G_{\lambda_i}(x)$$

Then equations of the curves are as in cases 24-31 in Table 4, where $Q(x) = \psi(x)$.

4.6. $\bar{G} \cong U$**.** The branch point of the curve ϕ is $\{\infty\}$. Let $q_1 = \infty$. Then $\varphi(x) = 1$. For each $\lambda_i \in S \backslash \{\infty\}$ we have

$$G_{\lambda_i}(x) = \prod_{a \in H_t} (x + a) - \lambda_i$$

There are δ points in $S \backslash \{\infty\}$. Where δ is as in Table 2. We denote

$$B(x) = \prod_{i=1}^{\delta} G_{\lambda_i}(x)$$

Then, each family is parameterized as cases 32-33.

4.7. $\bar{G} \cong K_m$**.** The branch points of the curve ϕ are $\{0, \infty\}$. Let $q_1 = 0$, $q_2 = \infty$. Then the polynomial over the branch point is

$$\varphi(x) = x \prod_{j=1}^{\frac{p^t-1}{m}} (x^m - b_j)$$
$$\chi(x) = 1$$

For $\lambda_i \in S \backslash \{0, \infty\}$ we have

$$G_{\lambda_i}(x) = ((x \prod_{j=1}^{\frac{p^t-1}{m}} (x^m - b_j))^m - \lambda_i)$$

There are δ points in $S \backslash \{0, \infty\}$. Where δ is as in Table 2. We denote

$$\Theta(x) = \prod_{i=1}^{\delta} G_{\lambda_i}(x)$$

Then, each family is parameterized as cases 34-37.

4.8. $\bar{G} \cong PSL_2(q)$**.** The branch points of $\phi(x)$ are $\{0, \infty\}$. Let $q_1 = 0$, $q_2 = \infty$. Then

$$\varphi(x) = (x^q - x)^{q-1} + 1$$
$$\chi(x) = x^q - x$$

For $\lambda_i \in S \backslash \{0, \infty\}$, points in $\phi^{-1}(\lambda_i)$ are roots of the polynomials,

$$G_{\lambda_i}(x) = (((x^q - x)^{q-1} + 1)^{\frac{q+1}{2}} - \lambda_i (x^q - x)^{\frac{q(q-1)}{2}})$$

There are δ points in $S \backslash \{0, \infty\}$. Where δ is as in Table 2. We denote

$$\Delta(x) = \prod_{i=1}^{\delta} (((x^q - x)^{q-1} + 1)^{\frac{q+1}{2}} - \lambda_i (x^q - x)^{\frac{q(q-1)}{2}})$$

Then, each family is parameterized as cases 38-41.

4.9. $\bar{G} \cong PGL_2(q)$**.** The branch points of $\phi(x)$ are $\{0, \infty\}$. Let $q_1 = 0$, $q_2 = \infty$. Then

$$\varphi(x) = (x^q - x)^{q-1} + 1$$
$$\chi(x) = x^q - x$$

For $\lambda_i \in S \backslash \{0, \infty\}$, points in $\phi^{-1}(\lambda_i)$ are roots of the polynomials,

$$G_{\lambda_i}(x) = (((x^q - x)^{q-1} + 1)^{q+1} - \lambda_i (x^q - x)^{q(q-1)})$$

Then we let,

$$\Omega(x) = \prod_{i=1}^{\delta} (((x^q - x)^{q-1} + 1)^{q+1} - \lambda_i (x^q - x)^{q(q-1)})$$

where δ is gives as Table 2. Then, each family is parameterized as cases 42-45. This completes the proof.

□

Remark 2. *By Remark 1, we know that A_5 has different ramification when $p = 3$. In this case $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ has signature $(6, 5)$. The branch points of $\phi(x)$ are ∞ and 0. Let $q_1 = \infty$ and $q_2 = 0$. By Lemma 1, we know that*

$$\phi(x) = \frac{(x^{10} - 1)^6}{(x(x^{10} + 2ix^5 + 1))^5}.$$

Then,

$$\varphi(x) = x(x^{10} + 2ix^5 + 1)$$
$$\chi(x) = x^{10} - 1$$

For $\lambda_j \in S \setminus \{0, \infty\}$, the points in $\phi^{-1}(\lambda_j)$ are roots of the polynomial

$$\begin{aligned} G_{\lambda_j}(x) = x^{60} &+ \lambda_j x^{55} - (6 + 10i\lambda_j)x^{50} + 35\lambda_j x^{45} + (15 + 40i\lambda_j)x^{40} + 30\lambda_j x^{35} \\ &- (20 - 68i\lambda_j)x^{30} + 30\lambda_j x^{25} + (15 + 40i\lambda_j)x^{20} + 35\lambda_j x^{15} \\ &- (6 + 10i\lambda_j)x^{10} - \lambda_j x^5 + 1 \end{aligned}$$

There are δ points in $S \setminus \{0, \infty\}$, where δ is given as in Table 3. We denote

$$P(x) := \prod_{j=1}^{\delta} G_{\lambda_j}(x)$$

Then, each family is parameterized as in Table 5.

Case	$y^n =$
a	$P(x)$
b	$x(x^{10} + 2ix^5 + 1) \cdot P(x)$
c	$(x^{10} - 1) \cdot P(x)$
d	$x(x^{10} + 2ix^5 + 1)(x^{10} - 1) \cdot P(x)$

TABLE 5. Equation of curve when $\bar{G} \cong A_5$, $p = 3$

Lemma 2. *Let $\mathcal{X}_g$ be a cyclic curve defined over an algebraically closed field k of characteristic $p = 3$ such that $\bar{G}$ for $\mathcal{X}_g$ is isomorphic to A_5. Then, the equation of $\mathcal{X}_g$ is as in one of the cases in Table 5.*

We summarize all the cases in the following Theorem.

Theorem 4.2. *Let $\mathcal{X}_g$ be e genus $g \geq 2$ algebraic curve defined over an algebraically closed field k, G its automorphism group over k, and H cyclic normal subgroup of G of order n such that $g(X_g^H) = 0$. Then, the equation of $\mathcal{X}_g$ can be written as in one of the following cases:*

#	G	$y^n = f(x)$
1		$x^{m\delta} + a_1 x^{m(\delta-1)} + \dots + a_\delta x^m + 1$
2	C_m	$x^{m\delta} + a_1 x^{m(\delta-1)} + \dots + a_\delta x^m + 1$
3		$x(x^{m\delta} + a_1 x^{m(\delta-1)} + \dots + a_\delta x^m + 1)$
4		$F(x) := \prod_{i=1}^{\delta}(x^{2m} + \lambda_i x^m + 1)$
5		$(x^m - 1) \cdot F(x)$
6		$x \cdot F(x)$
7	D_{2m}	$(x^{2m} - 1) \cdot F(x)$
8		$x(x^m - 1) \cdot F(x)$
9		$x(x^{2m} - 1) \cdot F(x)$
10		$G(x) := \prod_{i=1}^{\delta}(x^{12} - \lambda_i x^{10} - 33x^8 + 2\lambda_i x^6 - 33x^4 - \lambda_i x^2 + 1)$
11		$(x^4 + 2i\sqrt{3}x^2 + 1) \cdot G(x)$
12	A_4	$(x^8 + 14x^4 + 1) \cdot G(x)$
13		$x(x^4 - 1) \cdot G(x)$
14		$x(x^4 - 1)(x^4 + 2i\sqrt{3}x^2 + 1) \cdot G(x)$
15		$x(x^4 - 1)(x^8 + 14x^4 + 1) \cdot G(x)$
16		$M(x)$
17		$\left(x^8 + 14x^4 + 1\right) \cdot M(x)$
18		$x(x^4 - 1) \cdot M(x)$
19		$\left(x^8 + 14x^4 + 1\right) \cdot x(x^4 - 1) \cdot M(x)$
20	S_4	$\left(x^{12} - 33x^8 - 33x^4 + 1\right) \cdot M(x)$
21		$\left(x^{12} - 33x^8 - 33x^4 + 1\right) \cdot \left(x^8 + 14x^4 + 1\right) \cdot M(x)$
22		$\left(x^{12} - 33x^8 - 33x^4 + 1\right) \cdot x(x^4 - 1) \cdot M(x)$
23		$\left(x^{12} - 33x^8 - 33x^4 + 1\right) \cdot \left(x^8 + 14x^4 + 1\right) \cdot x(x^4 - 1)M(x)$
24		$\Lambda(x)$
25		$(x(x^{10} + 11x^5 - 1)) \cdot \Lambda(x)$
26		$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1)(x(x^{10} + 11x^5 - 1)) \cdot \Lambda(x)$
27		$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1) \cdot \Lambda(x)$
28	A_5	$Q(x) \cdot \Lambda(x)$
29		$x(x^{10} + 11x^5 - 1).\psi(x) \cdot \Lambda(x)$
30		$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1) \cdot \psi(x) \cdot \Lambda(x)$
31		$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1)(x(x^{10} + 11x^5 - 1)) \cdot \psi(x) \cdot \Lambda(x)$
32	U	$B(x)$
33		$B(x)$
34		$\Theta(x)$
35	K_m	$x \prod_{j=1}^{\frac{p^t-1}{m}} (x^m - b_j) \cdot \Theta(x)$
36		$\Theta(x)$
37		$x \prod_{j=1}^{\frac{p^t-1}{m}} (x^m - b_j) \cdot \Theta(x)$
38		$\Delta(x)$
39	$PSL_2(q)$	$((x^q - x)^{q-1} + 1) \cdot \Delta(x)$
40		$(x^q - x) \cdot \Delta(x)$
41		$(x^q - x)((x^q - x)^{q-1} + 1) \cdot \Delta(x)$
42		$\Omega(x)$
43	$PGL_2(q)$	$((x^q - x)^{q-1} + 1) \cdot \Omega(x)$
44		$(x^q - x) \cdot \Omega(x)$
45		$(x^q - x)((x^q - x)^{q-1} + 1) \cdot \Omega(x)$

TABLE 6. The equations of the curves related to the cases in Table 2

where δ is given as in Table 2 and M, Λ, Q, B, Δ, and Ω are as follows:

$$
\begin{aligned}
M =& \prod_{i=1}^{\delta} \big(x^{24} + \lambda_i x^{20} + (759 - 4\lambda_i)x^{16} + 2(3\lambda_i + 1228)x^{12} \\
& + (759 - 4\lambda_i)x^8 + \lambda_i x^4 + 1\big) \\
\Lambda =& \prod_{i=1}^{\delta} \big(-x^{60} + (684 - \lambda_i)x^{55} - (55\lambda_i + 157434)x^{50} - (1205\lambda_i - 12527460)x^{45} \\
& - (13090\lambda_i + 77460495)x^{40} + (130689144 - 69585\lambda_i)x^{35} \\
& + (33211924 - 134761\lambda_i)x^{30} + (69585\lambda_i - 130689144)x^{25} \\
& - (13090\lambda_i + 77460495)x^{20} - (12527460 - 1205\lambda_i)x^{15} \\
& -(157434 + 55\lambda_i)x^{10} + (\lambda_i - 684)x^5 - 1\big) \\
Q =& x^{30} + 522x^{25} - 10005x^{20} - 10005x^{10} - 522x^5 + 1 \\
B =& \prod_{i=1}^{\delta} \prod_{a \in H_t} ((x + a) - \lambda_i) \\
\Delta =& \prod_{i=1}^{\delta} (((x^q - x)^{q-1} + 1)^{\frac{q+1}{2}} - \lambda_i (x^q - x)^{\frac{q(q-1)}{2}}) \\
\Omega =& \prod_{i=1}^{\delta} (((x^q - x)^{q-1} + 1)^{q+1} - \lambda_i (x^q - x)^{q(q-1)})
\end{aligned}
$$

References

[1] R. Brandt and H. Stichtenoth, Die Automorphismengruppen hyperelliptischer Kurven, Man. Math 55 (1986), 83–92.

[2] E. Bujulance, J. Gamboa, and G. Gromadzki, The full automorphism groups of hyperelliptic Riemann surfaces, Manuscripta Math. 79 (1993), no. 3-4, 267–282.

[3] W. Baily, On the automorphism group of a generic curve of genus > 2. J. Math. Kyoto Univ. 1 1961/1962 101–108; correction, 325.

[4] Y. Demirbas, Automorphism groups of hyperelliptic curves of genus 3 in characteristic 2, Computational aspects of algebraic curves, T. Shaska (Edt), Lect. Notes in Comp., World Scientific, 2005.

[5] Magaard, K.; Shaska, T.; Shpectorov, S.; Völklein, H.; The locus of curves with prescribed automorphism group. Communications in arithmetic fundamental groups (Kyoto, 1999/2001). Sürikaisekikenkyüsho Kökyüroku No. 1267 (2002), 112–141.

[6] Miller, G. A.; Blichfeldt, H. F.; Dickson, L. E.; Theory and applications of finite groups. (English) 2. ed. XVII + 390 p. New York, Stechert. Published: 1938

[7] P. G. Henn, Die Automorphismengruppen der algebraischen Funktionenkörper vom Geschlecht 3, PhD thesis, University of Heidelberg, (1976).

[8] P. Roquette, Abschtzung der Automorphismenanzahl von Funktionenkrpern bei Primzahlcharakteristik. Math. Z. 117 1970 157–163.

[9] A. Kontogeorgis, The Group of Automorphisms of Cyclic Extensions of Rational Function Fields, J. Algebra 216(2) (1999), 665-706.

[10] C. R. Valentini and L. M. Madan, A Hauptsatz of L. E. Dickson and Artin-Scheier extension, J. Reine Angew. Math. 318 (1980), 156-177.

[11] T. Shaska, Subvarieties of the Hyperelliptic Moduli Determined by Group Actions, Serdica Math. J. 32 (2006), 355-374.

[12] T. Shaska, Some special families of hyperelliptic curves. J. Algebra Appl. 3, 1 (2004), 75-89.

[13] T. Shaska, Determining the automorphism groups of hyperelliptic curves. Proceeding of the 2003 International Symposium on Symbolic Algebraic Computation, ACM Press, 2003, 248-254.

[14] T. Shaska and J. Thompson, On the generic curve of genus 3. Affine algebraic geometry, 233–243, Contemp. Math., 369, Amer. Math. Soc., Providence, RI, 2005.

[15] T. Shaska and H. Völklein, Elliptic subfields and automorphisms of genus 2 function fields. Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000), 703–723, Springer, Berlin, 2004.

[16] E. Previato, T. Shaska, and S. Wijesiri, Thetanulls of cyclic curves of small genus, *Albanian J. Math.*, Special issue on computational algebraic geometry, **vol. 1**, Nr. 4, 2007, pg. 265-282.

[17] T. Shaska, Some open problems in computational algebraic geometry, *Albanian J. Math.*, Special issue on computational algebraic geometry, **vol 1**, Nr. 4, 2007, 309-321.

[18] T. Shaska and C. Shor, Codes over F_{p^2} and $F_p \times F_p$, lattices, and theta functions, *Advances in Coding Theory and Cryptology*, vol 3. (2007), pg. 70-80.

[19] T. Shaska and Q. Wang, On the automorphism groups of some AG-codes based on C_{ab} curves, Serdica Journal of Computing, 2007, vol. 1. pg. 193-206.

[20] T. Shaska and D. Sevilla, Hyperelliptic curves with reduced automorphism group A_5, *Appl. Algebra Engrg. Comm. Comput.*, (2007), vol. 1, pg. 3-20.

[21] J. Gutierrez and T. Shaska, Hyperelliptic curves with extra involutions, *LMS J. of Comput. Math.*, 8 (2005), 102-115.

[22] T. Shaska and S. Zheng, A Maple package for hyperelliptic curves, Ed. I. Kotsieras, Maple conference, 2005, pg. 161-175.

[23] J. Gutierrez, D. Sevilla and T. Shaska, Hyperelliptic curves of genus 3 with prescribed automorphism group, *Lect. Notes in Computing*, vol 13. (2005), pg. 201-225.

[24] T. Shaska, Genus 2 curves covering elliptic curves, a computational approach, *Lect. Notes in Computing*, vol 13. (2005), pg. 151-195.

[25] T. Shaska, Genus 2 fields with degree 3 elliptic subfields, *Forum. Math.*, vol. **16**, 2, pg. 263-280, 2004.

[26] T. Shaska, Computational algebra and algebraic curves, ACM, *SIGSAM Bulletin*, *Comm. Comp. Alg.*, Vol. **37**, No. 4, 117-124, 2003.

[27] T. Shaska, Computational aspects of hyperelliptic curves, Computer mathematics. Proceedings of the sixth Asian symposium (ASCM 2003), Beijing, China, April 17-19, 2003. River Edge, NJ: World Scientific. *Lect. Notes Ser. Comput.* 10, 248-257 (2003).

[28] T. Shaska, Determining the automorphism group of a hyperelliptic curve, *Proceedings of the 2003 International Symposium on Symbolic and Algebraic Computation*, ACM Press, pg. 248 - 254, 2003.

[29] R. Sanjeewa and T. Shaska, Automorphism groups of cyclic curves, (preprint)

[30] H. Stichtenoth, ber die Automorphismengruppe eines algebraischen Funktionenkrpers von Primzahlcharakteristik. I. Eine Abschtzung der Ordnung der Automorphismengruppe. Arch. Math. (Basel) 24 (1973) 527–544.