# DEGREE EVEN COVERINGS OF ELLIPTIC CURVES BY GENUS 2 CURVES

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Abstract. In this survey we study the genus 2 curves with $(n, n)$-split Jacobian for even $n$.

## 1. Introduction

Let $C$ be a genus 2 curve defined over an algebraically closed field $k$, of characteristic zero. Let $\psi : C \to E$ be a degree $n$ maximal covering (i.e. does not factor through an isogeny) to an elliptic curve $E$ defined over $k$. We say that $C$ has a *degree $n$ elliptic subcover*. Degree $n$ elliptic subcovers occur in pairs. Let $(E, E')$ be such a pair. It is well known that there is an isogeny of degree $n^2$ between the Jacobian $J_C$ of $C$ and the product $E \times E'$. The locus of such $C$, denoted by $\mathcal{L}_n$, is a 2-dimensional algebraic subvariety of the moduli space $\mathcal{M}_2$ of genus two curves and has been the focus of many papers in the last decade; see [?, ?, ?, ?, ?, ?, ?].

The space $\mathcal{L}_2$ was studied in Shaska/Völklein [?]. The space $\mathcal{L}_3$ was studied in [?] were an algebraic description was given as sublocus of $\mathcal{M}_2$. Lately the space $\mathcal{L}_5$ has been studied in detail in [?]. The case of even degree has been less studied even though there have been some attempts lately to compute some of the cases for $n = 4$; see [?]. In this survey we study the genus 2 curves with $(n, n)$-split Jacobian for small $n$. While such curves have been studied by many authors, our approach is simply computational.

## 2. Curves of genus 2 with split Jacobians

Most of the results of this section can be found in [?]. Let $C$ and $E$ be curves of genus 2 and 1, respectively. Both are smooth, projective curves defined over $k$, $char(k) = 0$. Let $\psi : C \longrightarrow E$ be a covering of degree $n$. From the Riemann-Hurwitz formula, $\sum_{P \in C} (e_\psi(P) - 1) = 2$ where $e_\psi(P)$ is the ramification index of points $P \in C$, under $\psi$. Thus, we have two points of ramification index 2 or one point of ramification index 3. The two points of ramification index 2 can be in the same fiber or in different fibers. Therefore, we have the following cases of the covering $\psi$:

**Case I:** There are $P_1$, $P_2 \in C$, such that $e_\psi(P_1) = e_\psi(P_2) = 2$, $\psi(P_1) \neq \psi(P_2)$, and $\forall P \in C \setminus \{P_1, P_2\}$, $e_\psi(P) = 1$.

**Case II:** There are $P_1$, $P_2 \in C$, such that $e_\psi(P_1) = e_\psi(P_2) = 2$, $\psi(P_1) = \psi(P_2)$, and $\forall P \in C \setminus \{P_1, P_2\}$, $e_\psi(P) = 1$.

**Case III:** There is $P_1 \in C$ such that $e_\psi(P_1) = 3$, and $\forall P \in C \setminus \{P_1\}$, $e_\psi(P) = 1$.

In case I (resp. II, III) the cover $\psi$ has 2 (resp. 1) branch points in E.

Denote the hyperelliptic involution of $C$ by $w$. We choose $\mathcal{O}$ in E such that $w$ restricted to $E$ is the hyperelliptic involution on $E$. We denote the restriction of $w$ on $E$ by $v$, $v(P) = -P$. Thus, $\psi \circ w = v \circ \psi$. E[2] denotes the group of 2-torsion points of the elliptic curve E, which are the points fixed by $v$. The proof of the following two lemmas is straightforward and will be omitted.

**Lemma 1.** *a) If $Q \in E$, then $\forall P \in \psi^{-1}(Q)$, $w(P) \in \psi^{-1}(-Q)$.*
*b) For all $P \in C$, $e_\psi(P) = e_\psi(w(P))$.*

Let $W$ be the set of points in C fixed by $w$. Every curve of genus 2 is given, up to isomorphism, by a binary sextic, so there are 6 points fixed by the hyperelliptic involution $w$, namely the Weierstrass points of $C$. The following lemma determines the distribution of the Weierstrass points in fibers of 2-torsion points.

**Lemma 2.** *The following hold:*

(1) $\psi(W) \subset E[2]$
(2) *If $n$ is an odd number then*
   *i)* $\psi(W) = E[2]$
   *ii) If $Q \in E[2]$ then $\#(\psi^{-1}(Q) \cap W) = 1 \mod (2)$*
(3) *If $n$ is an even number then for all $Q \in E[2]$, $\#(\psi^{-1}(Q) \cap W) = 0 \mod (2)$*

Let $\pi_C : C \longrightarrow \mathbb{P}^1$ and $\pi_E : E \longrightarrow \mathbb{P}^1$ be the natural degree 2 projections. The hyperelliptic involution permutes the points in the fibers of $\pi_C$ and $\pi_E$. The ramified points of $\pi_C$, $\pi_E$ are respectively points in $W$ and $E[2]$ and their ramification index is 2. There is $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ such that the diagram commutes.

(1)
$$
\begin{array}{ccc}
C & \xrightarrow{\pi_C} & \mathbb{P}^1 \\
\psi \downarrow & & \downarrow \phi \\
E & \xrightarrow{\pi_E} & \mathbb{P}^1
\end{array}
$$

Next, we will determine the ramification of induced coverings $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$. First we fix some notation. For a given branch point we will denote the ramification of points in its fiber as follows. Any point $P$ of ramification index $m$ is denoted by $(m)$. If there are $k$ such points then we write $(m)^k$. We omit writing symbols for unramified points, in other words $(1)^k$ will not be written. Ramification data between two branch points will be separated by commas. We denote by $\pi_E(E[2]) = \{q_1, \ldots, q_4\}$ and $\pi_C(W) = \{w_1, \ldots, w_6\}$.

2.0.1. *The Case When $n$ is Even.* Let us assume now that $deg(\psi) = n$ is an even number. The following theorem classifies the induced coverings in this case.

**Theorem 1.** *If $n$ is an even number then the generic case for $\psi : C \longrightarrow E$ induce the following three cases for $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$:*

**I:** $\left( (2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n}{2}}, (2) \right)$

**II:** $\left((2)^{\frac{n-4}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)\right)$

**III:** $\left((2)^{\frac{n-6}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)\right)$

*Each of the above cases has the following degenerations (two of the branch points collapse to one)*

**I:** (1) $\left((2)^{\frac{n}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n}{2}}\right)$

(2) $\left((2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (4)(2)^{\frac{n-6}{2}}, (2)^{\frac{n}{2}}\right)$

(3) $\left((2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (4)(2)^{\frac{n-4}{2}}\right)$

(4) $\left((3)(2)^{\frac{n-4}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n}{2}}\right)$

**II:** (1) $\left((2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}\right)$

(2) $\left((2)^{\frac{n-4}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}\right)$

(3) $\left((4)(2)^{\frac{n-8}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}\right)$

(4) $\left((2)^{\frac{n-4}{2}}, (4)(2)^{\frac{n-6}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}\right)$

(5) $\left((2)^{\frac{n-4}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n-4}{2}}, (2)^{\frac{n}{2}}\right)$

(6) $\left((3)(2)^{\frac{n-6}{2}}, (2)^{\frac{n-2}{2}}, (4)(2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}\right)$

(7) $\left((2)^{\frac{n-4}{2}}, (3)(2)^{\frac{n-4}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}\right)$

**III:** (1) $\left((2)^{\frac{n-4}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (4)(2)^{\frac{n}{2}}\right)$

(2) $\left((2)^{\frac{n-6}{2}}, (4)(2)^{\frac{n-4}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}\right)$

(3) $\left((2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (4)(2)^{\frac{n-10}{2}}\right)$

(4) $\left((3)(2)^{\frac{n-8}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}\right)$

*Proof.* We skip the details of the proof.                         □

**Remark 1.** *The case $n = 8$ is the first true generic case when all the subcases occur.*

### 2.1. Maximal coverings $\psi : C \longrightarrow E$.

Let $\psi_1 : C \longrightarrow E_1$ be a covering of degree $n$ from a curve of genus 2 to an elliptic curve. The covering $\psi_1 : C \longrightarrow E_1$ is called a **maximal covering** if it does not factor through a nontrivial isogeny. A map of algebraic curves $f : X \to Y$ induces maps between their Jacobians $f^* : J_Y \to J_X$ and $f_* : J_X \to J_Y$. When $f$ is maximal then $f^*$ is injective and $ker(f_*)$ is connected, see [**?**] for details.

Let $\psi_1 : C \longrightarrow E_1$ be a covering as above which is maximal. Then $\psi^*_1 : E_1 \to J_C$ is injective and the kernel of $\psi_{1,*} : J_C \to E_1$ is an elliptic curve which we denote by $E_2$; see [**?**]. For a fixed Weierstrass point $P \in C$, we can embed $C$ to its Jacobian via

(2)
$$i_P : C \longrightarrow J_C$$
$$x \to [(x) - (P)]$$

Let $g : E_2 \to J_C$ be the natural embedding of $E_2$ in $J_C$, then there exists $g_* : J_C \to E_2$. Define $\psi_2 = g_* \circ i_P : C \to E_2$. So we have the following exact sequence

$$0 \to E_2 \xrightarrow{g} J_C \xrightarrow{\psi_{1,*}} E_1 \to 0$$

The dual sequence is also exact

$$0 \to E_1 \xrightarrow{\psi_1^*} J_C \xrightarrow{g_*} E_2 \to 0$$

If $deg(\psi_1)$ is an odd number then the maximal covering $\psi_2 : C \to E_2$ is unique up to isomorphism of elliptic curves. If the cover $\psi_1 : C \longrightarrow E_1$ is given, and therefore $\phi_1$, we want to determine $\psi_2 : C \longrightarrow E_2$ and $\phi_2$. The study of the relation between the ramification structures of $\phi_1$ and $\phi_2$ provides information in this direction. The following lemma (see [**?**, pg. 160]) answers this question for the set of Weierstrass points $W = \{P_1, \ldots, P_6\}$ of C when the degree of the cover is odd.

**Lemma 3.** *Let $\psi_1 : C \longrightarrow E_1$, be maximal of degree n. Then, the map $\psi_2 : C \to E_2$ is a maximal covering of degree n. Moreover,*

i) *if n is odd and $\mathcal{O}_i \in E_i[2]$, $i = 1, 2$ are the places such that $\#(\psi_i^{-1}(\mathcal{O}_i) \cap W) = 3$, then $\psi_1^{-1}(\mathcal{O}_1) \cap W$ and $\psi_2^{-1}(\mathcal{O}_2) \cap W$ form a disjoint union of W.*
ii) *if n is even and $Q \in E[2]$, then $\# \left( \psi^{-1}(Q) \right) \cap W = 0$ or 2.*

The above lemma says that if $\psi$ is maximal of even degree then the corresponding induced covering can have only type **I** ramification, see Theorem **??**.

**Example 1.** Let $\psi : C \to E$ be a degree $n = 8$ maximal covering of the elliptic curve $E$ by a genus 2 curve $C$. Then, we have Type I covering as in previous theorem. Hence, the ramification is

$$\left( (2)^3, (2)^3, (2)^3, (2)^4, (2) \right)$$

This case is the first case which has all its subcases with ramifications as follows:

**i)** $\left( (2)^4, (2)^3, (2)^3, (2)^4 \right)$

**ii)** $\left( (2)^3, (2)^3, (4)(2), (2)^4 \right)$

**iii)** $\left( (2)^3, (2)^3, (2)^3, (4)(2)^2 \right)$

**iv)** $\left( (3)(2)^2, (2)^3, (2)^3, (2)^4 \right)$

The locus of genus 2 curves in the generic case is a 2-dimensional subvariety of the moduli space $\mathcal{M}_2$. It would be interesting to explicitly compute such subvariety since it is the first case which could give some clues to what happens in the general case for even degree.

## 3. THE LOCUS OF GENUS TWO CURVES WITH $(n, n)$ SPLIT JACOBIANS

In this section we will discuss the Hurwitz spaces of coverings with ramification as in the previous section and the Humbert spaces of discriminant $n^2$.

3.1. **Hurwitz spaces of covers** $\phi : \mathbb{P}^1 \to \mathbb{P}^1$. Two covers $f : X \to \mathbb{P}^1$ and $f' : X' \to \mathbb{P}^1$ are called **weakly equivalent** if there is a homeomorphism $h : X \to X'$ and an analytic automorphism $g$ of $\mathbb{P}^1$ (i.e., a Moebius transformation) such that $g \circ f = f' \circ h$. The covers $f$ and $f'$ are called **equivalent** if the above holds with $g = 1$.

Consider a cover $f : X \to \mathbb{P}^1$ of degree $n$, with branch points $p_1, ..., p_r \in \mathbb{P}^1$. Pick $p \in \mathbb{P}^1 \setminus \{p_1, ..., p_r\}$, and choose loops $\gamma_i$ around $p_i$ such that $\gamma_1, ..., \gamma_r$ is a

standard generating system of the fundamental group $\Gamma := \pi_1(\mathbb{P}^1 \setminus \{p_1, ..., p_r\}, p)$, in particular, we have $\gamma_1 \cdots \gamma_r = 1$. Such a system $\gamma_1, ..., \gamma_r$ is called a homotopy basis of $\mathbb{P}^1 \setminus \{p_1, ..., p_r\}$. The group $\Gamma$ acts on the fiber $f^{-1}(p)$ by path lifting, inducing a transitive subgroup $G$ of the symmetric group $S_n$ (determined by $f$ up to conjugacy in $S_n$). It is called the **monodromy group** of $f$. The images of $\gamma_1, ..., \gamma_r$ in $S_n$ form a tuple of permutations $\sigma = (\sigma_1, ..., \sigma_r)$ called a tuple of **branch cycles** of $f$.

We say a cover $f : X \to \mathbb{P}^1$ of degree $n$ is of type $\sigma$ if it has $\sigma$ as tuple of branch cycles relative to some homotopy basis of $\mathbb{P}^1$ minus the branch points of $f$. Let $\mathcal{H}_\sigma$ be the set of weak equivalence classes of covers of type $\sigma$. The **Hurwitz space** $\mathcal{H}_\sigma$ carries a natural structure of an quasiprojective variety.

We have $\mathcal{H}_\sigma = \mathcal{H}_\tau$ if and only if the tuples $\sigma$, $\tau$ are in the same **braid orbit** $\mathcal{O}_\tau = \mathcal{O}_\sigma$. In the case of the covers $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ from above, the corresponding braid orbit consists of all tuples in $S_n$ whose cycle type matches the ramification structure of $\phi$.

This and the genus of $\mathcal{H}_\sigma$ in the degenerate cases (see the following table) has been computed in GAP by the BRAID PACKAGE written by K. Magaard.

| deg | Case | cycle type of $\sigma$ | $\#(\mathcal{O}_\sigma)$ | $G$ | dim $\mathcal{H}_\sigma$ | genus of $\mathcal{H}_\sigma$ |
|---|---|---|---|---|---|---|
| 8 | | $(2^3, 2^3, 2^3, 2^4, 2)$ | 224 | $S_8$ | 2 | – |
| | 1 | $(2^4, 2^3, 2^3, 2^4)$ | 4 | 16 | 1 | 0 |
| | 2 | $(2^3, 2^3, (4)(2), 2^4)$ | 48 | $S_8$ | 1 | 4 |
| | 3 | $(2^3, 2^3, 2^3, (4)(2)^2)$ | 96 | $S_8$ | 1 | 16 |
| | 4 | $((3)2^2, 2^3, 2^3, 2^4)$ | 36 | $S_8$ | 1 | 4 |

TABLE 1. The length of braid orbits, the order of the group, and the genus of 1-dimensional subspaces for even degree maximal coverings.

As the reader can imagine even such computations are not easy for higher $n$. It is unclear what are the monodromy groups that appear in all the subcases and the formulas for the lengths of the braid orbits.

3.2. **Humbert surfaces.** Let $\mathcal{A}_2$ denote the moduli space of principally polarized abelian surfaces. It is well known that $\mathcal{A}_2$ is the quotient of the Siegel upper half space $\mathfrak{H}_2$ of symmetric complex $2 \times 2$ matrices with positive definite imaginary part by the action of the symplectic group $Sp_4(\mathbb{Z})$.

Let $\Delta$ be a fixed positive integer and $N_\Delta$ be the set of matrices $\tau = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathfrak{H}_2$ such that there exist nonzero integers $a, b, c, d, e$ with the following properties:

$$az_1 + bz_2 + cz_3 + d(z_2^2 - z_1 z_3) + e = 0$$
$$\Delta = b^2 - 4ac - 4de$$

(3)

The *Humbert surface* $\mathcal{H}_\Delta$ of discriminant $\Delta$ is called the image of $N_\Delta$ under the canonical map

$$\mathfrak{H}_2 \to \mathcal{A}_2 := Sp_4(\mathbb{Z}) \setminus \mathfrak{H}_2.$$

It is known that $\mathcal{H}_\Delta \neq \emptyset$ if and only if $\Delta > 0$ and $\Delta \equiv 0 \ or \ 1 \mod 4$. Humbert (1900) studied the zero loci in Eq. (**??**) and discovered certain relations between points in these spaces and certain plane configurations of six lines.

For a genus 2 curve $C$ defined over $\mathbb{C}$, $[C]$ belongs too $\mathcal{L}_n$ if and only if the isomorphism class $[J_C] \in \mathcal{A}_2$ of its (principally polarized) Jacobian $J_C$ belongs to the Humbert surface $\mathcal{H}_{n^2}$, viewed as a subset of the moduli space $\mathcal{A}_2$ of principally polarized abelian surfaces. There is a one to one correspondence between the points in $\mathcal{L}_n$ and points in $\mathcal{H}_{n^2}$. Thus, we have the map:

(4)
$$\mathcal{H}_\sigma \longrightarrow \mathcal{L}_n \longrightarrow \mathcal{H}_{n^2}$$
$$([f], (p_1, \ldots, p_r) \to [\mathcal{X}] \to [J_\mathcal{X}]$$

In particular, every point in $\mathcal{H}_{n^2}$ can be represented by an element of $\mathfrak{H}_2$ of the form

$$\tau = \begin{pmatrix} z_1 & \frac{1}{n} \\ \frac{1}{n} & z_2 \end{pmatrix}, \qquad z_1, \, z_2 \in \mathfrak{H}.$$

There have been many attempts to explicitly describe these Humbert surfaces. For some small discriminant this has been done by several authors; see [**?**], [**?**]. Geometric characterizations of such spaces for $\Delta = 4, 8, 9$, and $12$ were given by Humbert (1900) in [**?**] and for $\Delta = 13, 16, 17, 20, 21$ by Birkenhake/Wilhelm (2003).

## 4. Computing the locus $\mathcal{L}_n$ in $\mathcal{M}_2$

We take the most general case for maximal coverings of even degree, namely $n$, Type I. The ramification structure of $\phi : \mathbb{P}_x^1 \to \mathbb{P}_z^1$ is

$$\left( (2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n}{2}}, (2) \right)$$

We denote the branch points respectively $q_1, \ldots, q_5$. Let $q_1 = 0, q_2 = 1, q_3 = \infty$. The red places in $\mathbb{P}_x^1$ denote the unramified places and the black places all have ramification index 2. We pick the coordinate $x$ such that it is $x = 0, x = 1, x = \infty$ in the unramified places of $\mathbb{P}_z^1$ and respectively in the fibers of $0, 1, \infty$ as in the picture.

There are exactly $d = \frac{n-2}{2}$ places of index 2 in $\phi^{-1}(0)$. Let $P(x)$ denote the polynomial whose roots are exactly these places. Similarly denote by $R(x), Q(x)$ such polynomials for fibers of 1 and $\infty$. The other unramified places in the fibers of $0, 1, \infty$ we denote by $w_4, w_5, w_6$ respectively.
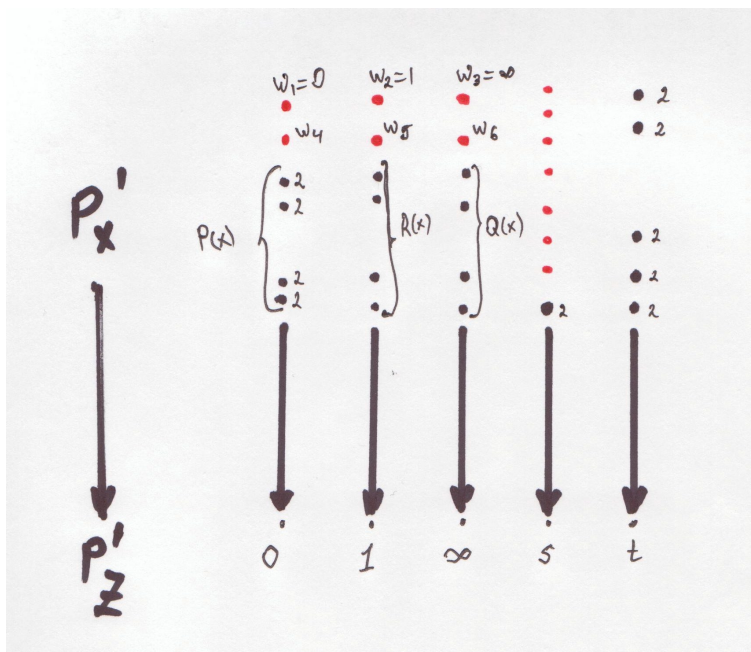
Then, we have

$$z = \lambda \cdot x \frac{x - w_4}{x - w_6} \cdot \frac{P^2(x)}{Q^2(x)}$$

for some $\lambda \cdot \in \mathbb{C}, \lambda \neq 0$. Furthermore,

$$z - 1 = \lambda \cdot (x - 1) \cdot \frac{x - w_5}{x - w_6} \cdot \frac{R^2(x)}{Q^2(x)}$$

where $P(x), Q(x), R(x)$ are monic polynomials of degree $d = \frac{n-2}{2}$ with no multiple roots and no common roots.

Substituting for $z$ we get a degree $n$ equation

$$\lambda x(x - w_4)P^2(x) - (x - w_6)Q^2(x) - \lambda \cdot (x - 1)(x - w_5)R^2(x) = 0$$

By equaling coefficients of this polynomial with zero we get a nonlinear system of $n + 1$ equations. In the same way we get the corresponding equations from the fibers of the other two branch points $s$ and $t$. Solving such system would determine also $w_4, w_5, w_6$. The equation of the genus 2 curve $C$ is given by

$$y^2 = x(x - 1)(x - w_4)(x - w_5)(x - w_6)$$

4.1. **Degree 4 covers.** In this section we focus on the case $\deg(\phi) = 4$ (not necessarily maximal). The goal is to determine all ramifications $\sigma$ and explicitly compute $\mathcal{L}_4(\sigma)$. There is one generic case and one degenerate case in which the ramification of $\deg(\phi) = 4$ applies, as given by the above possible ramification structures.

    i) $(2, 2, 2, 2^2, 2)$ (generic)
    ii) $(2, 2, 2, 4)$ (degenerate)

4.2. **Degenerate Case.** In this case one of the Weierstrass points has ramification index 3, so the cover is totally ramified at this point.

Let the branch points be 0, 1, $\lambda$, and $\infty$, where $\infty$ corresponds to the element of index 4. Then, above the fibers of 0, 1, $\lambda$ lie two Weierstrass points. The two Weierstrass points above 0 can be written as the roots of a quadratic polynomial $x^2 + ax + b$; above 1, they are the roots of $x^2 + px + q$; and above $\lambda$, they are the roots of $x^2 + sx + t$. This gives us an equation for the genus 2 curve $C$:

$$C : y^2 = (x^2 + ax + b)(x^2 + px + q)(x^2 + sx + t).$$

The four branch points of the cover $\phi$ are the 2-torsion points $E[2]$ of the elliptic curve $E$, allowing us to write the elliptic subcover as

$$E : y^2 = x(x - 1)(x - \lambda).$$

We have the following theorem:

**Theorem 2.** *Let $C$ be a genus 2 curve with a degree 4 degenerate elliptic subcover. Then $C$ is isomorphic to the curve given by*

$$C : y^2 = \left( \frac{1-b}{3} + \frac{2}{3}(1-b)x + x^2 \right) \left( \frac{1}{12}(b-4)b + \frac{1}{3}(b-4)x + x^2 \right)$$

(5)
$$\left( b - \frac{2}{3}(b+2)x + x^2 \right)$$

$$E : v^2 = u(u-1)\left( u - \frac{b^3(4-b)}{16(b-1)} \right)$$

*where the corresponding discriminants of the right sides must be non-zero. Hence,*

(6)
$$\Delta_C := b(b-4)(b-2)(b-1)(2+b) \neq 0$$

(7)
$$\Delta_E := \frac{(b-4)^2(b-2)^6 b^6 (b+2)^2}{65536(b-1)^4} \neq 0.$$

*and its invariants satisfy*

$$1541086152812576000\, J_2{}^2 J_4{}^2 - 22835312232360960000\, J_2 J_4 J_6 + 5009676947631\, J_2{}^6$$
$$- 8782271900467200000\, J_6{}^2 + 1176812184652746480\, J_2{}^4 J_4 + 12448207102988800000\, J_4{}^3$$
$$- 3715799948429529600\, J_2{}^3 J_6 = 0$$

(8)
$$186626560000\, J_2{}^2 J_4{}^4 + 1389621447673433587445760000\, J_{10}{}^2 + \frac{282429536481}{10^4}\, J_2{}^{10}$$
$$+ 619923800736\, J_2{}^6 J_4{}^2 - 25600000000\, J_4{}^5 - \frac{28249152375924}{100}\, J_2{}^8 J_4$$
$$+ 266576269949878792320\, J_2{}^5 J_{10} - 510202022400\, J_2{}^4 J_4{}^3$$
$$+ 693067624145203200000\, J_2 J_4{}^2 J_{10} + 1763516708182388736000\, J_2{}^3 J_4 J_{10} = 0.$$

*Proof.* See [**?**]. □

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