EQUATIONS FOR GENERALIZED SUPERELLIPTIC RIEMANN SURFACES

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Abstract. Let $\mathcal{X}$ be a closed Riemann surface, τ a conformal automorphism of order $n \geq 2$, and N be the normalizer of $\langle\tau\rangle$ in $G = \mathrm{Aut}(\mathcal{X})$. If τ is central in N, then $\mathcal{X}$ is called a generalized superelliptic curve of level $\mathbf{n}$. In this paoer, we determine parametric equations of all curves $\mathcal{X}$ and their corresponding genus.

1. Introduction

Superelliptic curves are generalization of hyperelliptic curves, when instead of the hyperelliptic involution the automorphism group of the curve posses a cyclic normal subgroup of order n. They are smooth algebraic curves with affine equation $y^n = f(x)$, where $f(x)$ has nonzero discriminant. In many ways such curves are natural generalization of the hyperelliptic case and provide the most natural way of extending the theory of hyperelliptic curve to the general case; see [7] for a detailed account of this approach.

In [4] we introduced generalized superelliptic curves, which are curves where $f(x)$ can have multiple roots. More precisely, a conformal automorphism τ, of order $n \geq 2$, of a closed Riemann surface $\mathcal{X}$, of genus $g \geq 2$, which is central in $G = \mathrm{Aut}(\mathcal{X})$ and such that $\mathcal{X}/\langle\tau\rangle$ has genus zero, is called a superelliptic automorphism of level n. If $n = 2$, then τ is called hyperelliptic involution and it is known to be unique.

In [4], for the case $n \geq 3$, we proved that for any two superelliptic automorphisms τ_1 and τ_2 of level n of $\mathcal{X}$ if n is odd, then $\langle\tau_1\rangle = \langle\tau_2\rangle$. In the case that n is even, then the same uniqueness result holds, up to some explicit exceptional cases.

Let N be the normalizer of $\langle\tau\rangle$ in G. For $n \geq 3$, τ does not need to be central in N and, if it is central in N, it might be that $N \neq G$.

We follow the terminology in [3, 8] and [4]. If τ is central in G, then we called it is a **superelliptic automorphism of level n**, we also say that $H = \langle\tau\rangle$ is a **superelliptic group of level n** and that $\mathcal{X}$ is a **superelliptic curve of level n**.

If τ is central in N, then we called it is a **generalized superelliptic automorphism of level n**; we also say that $H = \langle\tau\rangle$ is a **generalized superelliptic group of level n**, and that $\mathcal{X}$ is a **generalized superelliptic curve of level n**.

The goal of this paper is to determine parametric equations of all generalized superelliptic curves of level **n**. Our main result is Theorem 2 where we determine the parametric equation and the corresponding genus of the curve in each case.

2010 *Mathematics Subject Classification.* 14H37; 14H45; 30F10.

Key words and phrases. generalized superelliptic curves, cyclic gonal curves, automorphisms, Riemann surfaces.

2. Preliminaries

Let $\mathcal{X}$ be a closed Riemann surface of genus $g \geq 2$ and let $G = \mathrm{Aut}\,(\mathcal{X})$ be its group of conformal automorphisms. If $\tau \in G$ has order $n \geq 2$ and $\mathcal{X}/\langle\tau\rangle$ has genus zero, then it is called n-**gonal**. In this case, we also say that $\langle\tau\rangle \cong C_n$ is a n-**gonal group** and that $\mathcal{X}$ is a **cyclic** n-**gonal** Riemann surface. A 2-gonal automorphism, also called a hyperelliptic involution, is known to be unique in G; in particular, it is central in G.

If $n \geq 3$ is a prime integer and $s \geq 3$ is the number of fixed points of τ, then $\langle\tau\rangle$ is known to be the unique n-Sylow subgroup of $\mathcal{X}$ if either $2n < s$ or $n \geq 5s - 7$.

For a general $n \geq 3$, under the assumption that the fixed points of each non-trivial power of τ is also fixed by τ, the uniqueness of $\langle\tau\rangle$ is also true under the assumption that $g > (n-1)^2$ [6]; see [1] and [4]. In this last case, the computation of G has been done in [9].

Let N be the normalizer of $\langle\tau\rangle$ in G. If $n = 2$ (the hyperelliptic case), then τ is central in $N = G$. For $n \geq 3$, τ does not need to be central in N and, if it is central in N, it might be that $N \neq G$. It follows from the results in [10], that generically τ is central in G.

If τ is central in G (respectively, central in N), then we called it a **superelliptic automorphism of level n** (respectively, **generalized superelliptic automorphism of level n**); we also say that $H = \langle\tau\rangle$ is a **superelliptic group of level n** (respectively, **generalized superelliptic group of level n**), and that $\mathcal{X}$ is a **superelliptic curve of level n** (respectively, **generalized superelliptic curve of level n**).

Let us consider a pair $(\mathcal{X}, \tau)$, where τ is a n-gonal automorphism of $\mathcal{X}$, and $H = \langle\tau\rangle \cong C_n$. An algebraic model for $(\mathcal{X}, \tau)$ can be constructed as follows.

Let us consider a Galois branched covering $\pi : \mathcal{X} \to \widehat{\mathbb{C}}$, whose deck covering group is $H = \langle\tau\rangle$, and let $p_1, \ldots, p_s \in \widehat{\mathbb{C}}$ be its branch values. Then there are integers

$$l_1, \ldots, l_s \in \{1, \ldots, n-1\}$$

satisfying that $l_1 + \cdots + l_s$ is a multiple of n and $\gcd(n, l_1, \ldots, l_s) = 1$, such that $\mathcal{X}$ can be described by an affine irreducible algebraic curve (which might have singularities) of the following form (called a **cyclic** n-**gonal curve**)

$$y^n = \prod_{j=1}^{s} (x - p_j)^{l_j}, \tag{1}$$

If one of the branch values is ∞, say $p_s = \infty$, then we need to delete the factor $(x - p_s)^{l_s}$ from the above equation. In this algebraic model, τ and π are given by

$$\tau(x, y) = (x, \omega_n y),$$

where $\omega_n = e^{2\pi i/n}$, and $\pi(x, y) = x$.

Theorem 1. *Let $\mathcal{X}$ be a cyclic n-gonal Riemann surface, described by the cyclic n-gonal curve Eq. (1), and N be the normalizer of*

$$H = \langle \tau(x, y) = (x, \omega_n y) \rangle$$

in $\mathrm{Aut}\,(\mathcal{X})$. Let $\theta : N \to \overline{N} = N/H$ the canonical projection homomorphism. Then τ is a generalized superelliptic automorphism of level n if and only if for all p_j and p_i in the same $\theta(N)$-orbits it holds that $l_j = l_i$.

2.1. **The finite groups of Möbius transformations.** Up to $\mathrm{PSL}_2(\mathbb{C})$-conjugation, the finite subgroups of the group $\mathrm{PSL}_2(\mathbb{C})$ of Möbius transformations are given by (see, for instance, [2])

$$\begin{aligned} C_m &:= \langle a(x) = \omega_m x \rangle, \\ D_m &:= \left\langle a(x) = \omega_m x, b(x) = \tfrac{1}{x} \right\rangle, \\ A_4 &:= \left\langle a(x) = -x, b(x) = \tfrac{i-x}{i+x} \right\rangle, \\ S_4 &:= \left\langle a(x) = ix, b(x) = \tfrac{i-x}{i+x} \right\rangle, \\ A_5 &:= \left\langle a(x) = \omega_5 x, b(x) = \tfrac{(1-\omega_5^4)x + (\omega_5^4 - \omega_5)}{(\omega_5 - \omega_5^3)x + (\omega_5^2 - \omega_5^3)} \right\rangle, \end{aligned} \tag{2}$$

where ω_m is a primitive m-th root of unity. For each of the above finite groups A, a Galois branched covering

$$f_A : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}},$$

with deck group A, is given as follows

$$\begin{array}{lcll} f_{C_m}(x) & = & x^m; & \text{branching: } (m,m). \\ f_{D_m}(x) & = & x^m + x^{-m}; & \text{branching: } (2,2,m). \\ f_{A_4}(x) & = & \dfrac{(x^4 - 2i\sqrt{3}x^2 + 1)^3}{-12i\sqrt{3}x^2(x^4-1)^2}; & \text{branching: } (2,3,3). \\ f_{S_4}(x) & = & \dfrac{(x^8 + 14x^4 + 1)^3}{108x^4(x^4-1)^4}; & \text{branching: } (2,3,4). \\ f_{A_5}(x) & = & \dfrac{(-x^{20} + 228x^{15} - 494x^{10} - 228x^5 - 1)^3}{1728x^5(x^{10} + 11x^5 - 1)^5}; & \text{branching: } (2,3,5), \end{array}$$

see [5]. In the above, the branching corresponds to the tuple of branch orders of the cone points of the orbifold $\widehat{\mathbb{C}}/A$.

3. Equations for generalized superelliptic curves

Let $\mathcal{X}$ be a generalized superelliptic curve of level n and $\tau \in G = \mathrm{Aut}(\mathcal{X})$ be a generalized superelliptic automorphism of level n (so, it is central in its normalizer N). We proceed to describe explicit algebraic equations for $\mathcal{X}$ and also explicit generators for N, by making a subtle modification of the classical method done by Horiuchi in [5] for the hyperelliptic situation.

Let $\pi : \mathcal{X} \to \widehat{\mathbb{C}}$ be a Galois branched cover with deck group $H = \langle \tau \rangle$ and let

$$\mathcal{B}_\pi = \{p_1, \ldots, p_s\} \subset \widehat{\mathbb{C}}$$

be its set of branch values. Let $\theta : N \to \overline{N}$ be the surjective homomorphism satisfying $\theta(\eta) \circ \pi = \pi \circ \eta$, for every $\eta \in N$. Recall that $\overline{N}$ is one of the finite subgroups of $\mathrm{PSL}_2(\mathbb{C})$ (as described in Section 2.1) keeping the set $\mathcal{B}_\pi$ invariant.

Let us consider the Galois branched cover $f = f_{\overline{N}} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\overline{N}$ as its deck group (as described in Section 2.1). Let $P(x), Q(x) \in \mathbb{C}[x]$ be relatively prime polynomials such that $f(x) = \frac{P(x)}{Q(x)}$.

The collection $\mathcal{B}_\pi$ is $\overline{N}$-invariant and, by Theorem 1, if for $t \in \overline{N}$ it holds that $t(p_i) = p_j$, then $l_i = l_j$. In particular, we may consider the partition

$$\mathcal{B}_\pi = \mathcal{B}_\pi^{crit} \cup \mathcal{B}_\pi^{*},$$

where $\mathcal{B}_\pi^{crit}$ consists of those branch values with non-trivial $\overline{N}$-stabilizer. For simplicity, we assume $\infty \notin \mathcal{B}_\pi^{*}$ (but it might happen that $\infty \in \mathcal{B}_\pi^{crit}$).

3.1. **Horiuchi's general process.**

3.1.1. *Computing algebraic models.* There is at most $T \leq 3$ disjoint $\overline{N}$-orbits of the points in $\mathcal{B}_{\pi}^{crit}$.

i) If $\overline{N} \cong C_m$, then $T \leq 2$; each such orbit has cardinality one.
ii) If $\overline{N} \cong D_m$, then $T \leq 3$; at most one orbit of cardinality 2 and at most two others, each of cardinality m.
iii) If $\overline{N} \cong A_4$, then $T \leq 3$; at most one orbit of cardinality 6 and at most two others, each of cardinality 4.
iv) If $\overline{N} \cong S_4$, then $T \leq 3$; at most one orbit of cardinality 8, one of cardinality 6 and another of cardinality 12.
v) If $\overline{N} \cong A_5$, then $T \leq 3$; at most one orbit of cardinality 20, one of cardinality 30 and another of cardinality 12.

Let us denote these orbits (eliminating ∞ from its orbit if it is a branch value of π) by

$$\mathcal{O}_u^{crit} = \{q_{u,1}, \dots, q_{u,s_u}\}, \; u = 1, \dots, T, \tag{3}$$

where $s = s_1 + \cdots + s_T$ is the cardinality of $\mathcal{B}_{\pi}^{crit}$ if $\infty \notin \mathcal{B}_{\pi}^{crit}$ (otherwise, this cardinality is $s+1$).

Similarly, let the disjoint $\overline{N}$-orbits of the points in $\mathcal{B}_{\pi}^{*}$ be given by

$$\mathcal{O}_k^{*} = \{p_{k,1}, \dots, p_{k,|\overline{N}|}\}, \; k = 1, \dots, L, \tag{4}$$

(so, $L|\overline{N}|$ is the cardinality of $\mathcal{B}_{\pi}^{*}$).

As, for $k = 1, \dots, L$,

$$\prod_{j=1}^{|\overline{N}|}(x - p_{k,j}) = P(x) - f(p_{k,1})Q(x),$$

our curve can be written as

$$\mathcal{X}: \quad y^n = \prod_{u=1}^{T} R_u(x)^{\widehat{l}_u} \prod_{k=1}^{L} \left(P(x) - f(p_{k,1})Q(x)\right)^{\widetilde{l}_k}, \tag{5}$$

where

(1) $R_u(x) = \prod_{j=1}^{s_u}(x - q_{u,j})$
(2) $\widehat{l}_u \in \{0, 1, \dots, n-1\}$ and $\widetilde{l}_k \in \{1, \dots, n-1\}$.
(3) $\gcd(n, \widehat{l}_1, \dots \widehat{l}_T, \widetilde{l}_1, \dots, \widetilde{l}_L) = 1$ (where we eliminate a zero if appears).
(4) if $\infty \notin \mathcal{B}_{\pi}^{crit}$, then

$$\sum_{u=1}^{T} s_u \widehat{l}_u + \sum_{k=1}^{L} |\overline{N}| \widetilde{l}_k \equiv 0 \mod n.$$

(5) If $\infty \in \mathcal{O}_v^{crit}$, then

$$(1 + s_v)\widehat{l}_v + \sum_{u=1, u \neq v}^{T} s_u \widehat{l}_u + \sum_{k=1}^{L} |\overline{N}| \widetilde{l}_k \equiv 0 \mod n.$$

3.1.2. *Computing the elements of* N. Let $\eta \in N$ and $b = \theta(\eta)$. As τ commutes with η, by [4, Lem. 1],

$$\eta(x, y) = (b(x), F(x)y),$$

where $F(x) \in \mathbb{C}(x)$. Below, we sketch how to compute such $F(x)$.

Lemma 1. *Let* $\mathcal{O} = \{a_1, \ldots, a_r\}$ *a full* $\overline{N}$*-orbit (in our case, this is one of the* $\mathcal{O}_u^{crit}$ *or* $\mathcal{O}_k^*$*). If* $b \in \overline{N}$*, then the following hold.*

(1) If $\infty \notin \mathcal{O}$*, then*

$$\prod_{j=1}^{r}(b(x) - a_j) = (b'(x))^{r/2} \left(\prod_{j=1}^{r} b'(a_j)\right)^{1/2} \prod_{j=1}^{r}(x - a_j).$$

(2) If $a_r = \infty$ *and* $b(\infty) = \infty$*, then*

$$\prod_{j=1}^{r-1}(b(x) - a_j) = (b'(x))^{(r-1)/2} \left(\prod_{j=1}^{r-1} b'(a_j)\right)^{1/2} \prod_{j=1}^{r-1}(x - a_j).$$

(3) If $a_r = \infty$*,* $b(a_{r-1}) = \infty$ *and* $b(\infty) = a_s$*, where* $s \neq r - 1$*, and* $b(a_t) = a_{r-1}$*, then*

$$\prod_{j=1}^{r-1}(b(x) - a_j) = \frac{(a_{r-1} - a_s)^{\frac{1}{2}}(a_{r-1} - a_t)^{\frac{1}{2}}}{x - a_{r-1}} (b'(x))^{\frac{r-1}{2}} \left(\prod_{j=1}^{r-2} b'(a_j)\right)^{\frac{1}{2}} \prod_{j=1}^{r-1}(x - a_j).$$

(4) If $a_r = \infty$*,* $b(a_{r-1}) = \infty$ *and* $b(\infty) = a_{r-1}$*, then*

$$\prod_{j=1}^{r-1}(b(x) - a_j) = -(b'(x))^{r/2} \left(\prod_{j=1}^{r-2} b'(a_j)\right)^{1/2} \prod_{j=1}^{r-1}(x - a_j).$$

Proof. The equalites are consequence of the fact that, for $a, b(a) \in \mathbb{C}$,

$$b(x) - b(a) = b'(x)^{1/2} b'(a)^{1/2}(x - a).$$

□

If, in the above lemma, we replace $\mathcal{O}$ by $\mathcal{O}_u^{crit}$, then we obtain an equality

$$R_u(b(x)) = \prod_{j=1}^{s_u}(b(x) - q_{u,j}) = Q_u(x) \prod_{j=1}^{s_u}(x - q_{u,j}) = Q_u(x)R_u(x).$$

Similarly, if we replace $\mathcal{O}$ by $\mathcal{O}_k^*$ and set

$$S_k(x) := P(b(x)) - f(p_{k,1})Q(b(x)) = \prod_{j=1}^{|\overline{N}|}(x - p_{k,j}),$$

then we obtain an equality

$$S_k(b(x)) = \prod_{j=1}^{|\overline{N}|}(b(x) - p_{k,j}) = L_k(x) \prod_{j=1}^{|\overline{N}|}(x - p_{k,j}) = L_k(x)S_k(x).$$

It can be checked, by plugging directly into the equation for $\mathcal{X}$, that

$$F(x)^n = \prod_{u=1}^{T} Q_u(x)^{\widehat{l}_u} \prod_{k=1}^{L} L_k(x)^{\widetilde{l}_k}. \tag{6}$$

3.2. **Explicit computations.** Below, for each of the possibilities for $\overline{N}$, we proceed to explicitly describe the above procedure. In the following, if $l_u > 0$, then we set $n_u = \gcd(n, l_u)$.

Theorem 2. *Let $\mathcal{X}$ be a generalized superelliptic curve of level n,*

$$\tau \in G = \mathrm{Aut}\,(\mathcal{X})$$

be a generalized superelliptic automorphism of order n and N be the normalizer of $H = \langle \tau \rangle$ in G. Then, up to isomorphisms, $\mathcal{X}$, τ and N are described as indicated in the above cases.

$\bar{N}$	*Equation*	*Genus*
C_m	*Eq.* (7)	*Eq.* (8)
D_m	*Eq.* (9)	*Eq.* (10)
A_4	*Eq.* (11)	*Eq.* (12)
S_4	*Eq.* (13)	*Eq.* (14)
A_5	*Eq.* (15)	*Eq.* (16)

TABLE 1. The equations and the corresponding genii for each case

Proof. We will consider all cases one by one.

Case $\overline{\mathbf{N}} \cong \mathbf{C_m}$: In this case, $\overline{N} = \left\langle a(x) = \omega_m x \right\rangle$ and the curve $\mathcal{X}$ has the form

$$\mathcal{X}: \quad y^n = x^{l_0}(x^m - 1)^{l_1} \textstyle\prod_{j=2}^{r} (x^m - a_j^m)^{l_j}, \tag{7}$$

where

(i) $a_2, \dots, a_r \in \mathbb{C} - \{0, 1\}$, $a_i^m \neq a_j^m$ and
(ii) $\gcd(n, l_0, l_1, \dots, l_r) = 1$.

If

$$\alpha(x, y) = (\omega_m x, \omega_m^{l_0/n} y),$$

then

$$N = \langle \tau, \alpha : \tau^n = 1, \alpha^m = \tau^{l_0}, \tau\alpha = \alpha\tau \rangle.$$

The signature of $\mathcal{X}/H$ is

$$\begin{cases} \left(0; \frac{n}{n_1}, \overset{m}{\dots}, \frac{n}{n_1}, \dots, \frac{n}{n_r}, \overset{m}{\dots}, \frac{n}{n_r}\right), & \text{if } l_0 = 0,\ m\sum_{j=1}^{r} l_j \equiv 0 \mod (n), \\ \left(0; \frac{n}{n_0}, \frac{n}{n_1}, \overset{m}{\dots}, \frac{n}{n_1}, \dots, \frac{n}{n_r}, \overset{m}{\dots}, \frac{n}{n_r}\right), & \text{if } l_0 \neq 0,\ l_0 + m\sum_{j=1}^{r} l_j \equiv 0 \mod (n), \\ \left(0; \frac{n}{n_0}, \frac{n}{n_{r+1}}, \frac{n}{n_1}, \overset{m}{\dots}, \frac{n}{n_1}, \dots, \frac{n}{n_r}, \overset{m}{\dots}, \frac{n}{n_r}\right), & \text{if } l_0 \neq 0,\ l_0 + m\sum_{j=1}^{r} l_j \not\equiv 0 \mod (n), \end{cases}$$

where (in the last situation) $l_{r+1} \in \{1,\dots,n-1\}$ is the class of $-(l_0 + m\sum_{j=1}^{r} l_j)$ module n. The signature of $\mathcal{X}/N$ is

$$\begin{cases} \left(0; m, m, \frac{n}{n_1}, \frac{n}{n_2}, \dots, \frac{n}{n_r}\right), & \text{if } l_0 = 0,\ m\sum_{j=1}^{r} l_j \equiv 0 \mod (n), \\ \left(0; m, \frac{mn}{n_0}, \frac{n}{n_1}, \frac{n}{n_2}, \dots, \frac{n}{n_r}\right), & \text{if } l_0 \neq 0,\ l_0 + m\sum_{j=1}^{r} l_j \equiv 0 \mod (n), \\ \left(0; \frac{mn}{n_0}, \frac{mn}{n_{r+1}}, \frac{n}{n_1}, \frac{n}{n_2}, \dots, \frac{n}{n_r}\right), & \text{if } l_0 \neq 0,\ l_0 + m\sum_{j=1}^{r} l_j \not\equiv 0 \mod (n), \end{cases}$$

The genus of $\mathcal{X}$ is

(8)
$$\begin{cases} 1 + \frac{1}{2}\left((rm-2)n - m\sum_{j=1}^{r} n_j\right), & \text{if } l_0 = 0,\ m\sum_{j=1}^{r} l_j \equiv 0 \mod (n), \\ 1 + \frac{1}{2}\left((rm-1)n - m\sum_{j=1}^{r} n_j\right), & \text{if } l_0 \neq 0, l_0 + m\sum_{j=1}^{r} l_j \equiv 0 \mod (n), \\ 1 + \frac{1}{2}\left(rmn - m\sum_{j=1}^{r} n_j\right), & \text{if } l_0 \neq 0, l_0 + m\sum_{j=1}^{r} l_j \not\equiv 0 \mod (n). \end{cases}$$

Case $\overline{\mathbf{N}} \cong \mathbf{D_m}$: In this case, $D_m := \left\langle a(x) = \omega_m x, b(x) = \frac{1}{x} \right\rangle$ and the curve $\mathcal{X}$ has the form

$$\mathcal{X}: \quad y^n = x^{l_0}(x^m-1)^{l_{r+1}}(x^m+1)^{l_{r+2}} \prod_{j=1}^{r} (x^{2m} - (a_j^m + a_j^{-m})x^m + 1)^{l_j}, \tag{9}$$

where

(i) $a_i^{\pm m} \neq a_j^{\pm m} \neq 0, \pm 1$,
(ii) $2l_0 + m(l_{r+1} + l_{r+2}) + 2m(l_1 + \cdots + l_r) \equiv 0 \mod (n)$
(iii) $\gcd(n, l_0, l_1, \dots, l_{r+2}) = 1$.

If α and β are as follows

$$\alpha(x,y) = (\omega_m x, \omega_m^{l_0/n} y),$$
$$\beta(x,y) = \left(\frac{1}{x}, \frac{(-1)^{l_{r+1}/n}}{x^{(2l_0 + m(l_{r+1} + l_{r+2} + 2(l_1 + \cdots + l_r)))/n}} y\right),$$

then

$$N = \langle \tau, \alpha, \beta : \tau^n = 1, \alpha^m = \tau^{l_0},\ \beta^2 = \tau^{l_{r+1}},\ \tau\alpha = \alpha\tau,\ \tau\beta = \beta\tau \rangle.$$

The signature of $\mathcal{X}/H$ is

$$\left(0; \frac{n}{n_0}, \frac{n}{n_0}, \frac{n}{n_{r+1}}, \overset{m}{\dots}, \frac{n}{n_{r+1}}, \frac{n}{n_{r+2}}, \overset{m}{\dots}, \frac{n}{n_{r+2}}, \frac{n}{n_1}, \overset{2m}{\dots}, \frac{n}{n_1}, \dots, \frac{n}{n_r}, \overset{2m}{\dots}, \frac{n}{n_r}\right),$$

the signature of $\mathcal{X}/N$ is

$$\left(0; \frac{mn}{n_0}, \frac{2n}{n_{r+1}}, \frac{2n}{n_{r+2}}, \frac{n}{n_1}, \frac{n}{n_2}, \dots, \frac{n}{n_r}\right),$$

and the genus of $\mathcal{X}$ is

$$g = 1 + \frac{1}{2}\left(2m(r+1)n - 2n_0 - m\left(n_{r+1} + n_{r+2} + 2\sum_{j=1}^{r} n_j\right)\right). \tag{10}$$

Case $\overline{\mathbf{N}} \cong \mathbf{A_4}$: In this case, $A_4 := \left\langle a(x) = -x, b(x) = \frac{i-x}{i+x} \right\rangle$ and $\mathcal{X}$ has the form

(11)
$$\mathcal{X}: \ y^n = R_1(x)^{l_{r+1}} R_2(x)^{l_{r+2}} R_3(x)^{l_{r+3}} \prod_{j=1}^{r} \left(R_1(x)^3 + 12 i b_j \sqrt{3} R_3(x)^2\right)^{l_j},$$

where

$$\begin{aligned}R_1(x) &= x^4 - 2i\sqrt{3}x^2 + 1,\\ R_2(x) &= x^4 + 2i\sqrt{3}x^2 + 1,\\ R_3(x) &= x(x^4-1),\\ f(x) &= \frac{R_1(x)^3}{-12i\sqrt{3}\,R_3(x)^2},\end{aligned}$$

such that

(i) $b_j \neq b_i \in \mathbb{C}\setminus\{0,1\}$,
(ii) $4(l_{r+1}+l_{r+2}) + 6l_{r+3} + 12(l_1+\cdots+l_r) \equiv 0 \mod (n)$, and
(iii) $\gcd(n, l_1, \ldots, l_{r+3}) = 1$.

If

$$\alpha(x,y) = (-x, (-1)^{l_{r+3}/n}y), \quad \beta(x,y) = (b(x), F(x)y),$$

where

$$F(x) = \frac{2^{(l_{r+1}+l_{r+2})/n}(1-I\sqrt{3})^{l_{r+1}/n}(1+I\sqrt{3})^{l_{r+2}/n}(8i)^{l_{r+3}/n}(-64)^{(l_1+\cdots+l_r)/n}}{(x+i)^{(4(l_{r+1}+l_{r+2})+6l_{r+3}+12(l_1+\cdots+l_r))/n}},$$

then

$$\begin{aligned}N = &\langle \tau, \alpha, \beta :\ \tau^n = 1, \alpha^2 = \tau^{l_{r+3}},\ \beta^3 = \tau^{l_{r+1}+l_{r+2}+l_{r+3}+l_1+\cdots+l_r},\\ &(\alpha\beta)^3 = \tau^{l_{r+1}+l_{r+2}+3l_{r+3}+l_1+\cdots+l_r}, \tau\alpha = \alpha\tau,\ \tau\beta = \beta\tau\rangle.\end{aligned}$$

The signature of $\mathcal{X}/H$ is

$$\left(0; \frac{n}{n_{r+1}}, \overset{4}{\ldots}, \frac{n}{n_{r+1}}, \frac{n}{n_{r+2}}, \overset{4}{\ldots}, \frac{n}{n_{r+2}}, \frac{n}{n_{r+3}}, \overset{6}{\ldots}, \frac{n}{n_{r+3}}, \frac{n}{n_1}, \overset{12}{\ldots}, \frac{n}{n_1}, \ldots, \frac{n}{n_r}, \overset{12}{\ldots}, \frac{n}{n_r}\right),$$

the signature of $\mathcal{X}/N$ is

$$\left(0; \frac{3n}{n_{r+1}}, \frac{3n}{n_{r+2}}, \frac{2n}{n_{r+3}}, \frac{n}{n_1}, \frac{n}{n_2}, \ldots, \frac{n}{n_r}\right),$$

and the genus of $\mathcal{X}$ is

$$g = 1 - n + 2n_{r+1} + 2n_{r+2} + 3n_{r+2} - 6\sum_{j=1}^{r} n_j. \tag{12}$$

Case $\overline{\mathbf{N}} \cong \mathbf{S_4}$: In this case, $S_4 := \left\langle a(x) = ix, b(x) = \frac{i-x}{i+x}\right\rangle$ and $\mathcal{X}$ has the form

$$\mathcal{X}:\quad y^n = R_1(x)^{l_{r+1}}R_2(x)^{l_{r+2}}R_3(x)^{l_{r+3}}\prod_{j=1}^{r}(R_1(x)^3 - 108b_jR_3(x)^4)^{l_j}, \tag{13}$$

where

$$\begin{aligned}R_1(x) &= x^8 + 14x^4 + 1,\\ R_2(x) &= x^{12} - 33x^8 - 33x^4 + 1,\\ R_3(x) &= x(x^4-1),\\ f(x) &= \tfrac{R_1(x)^3}{108R_3(x)^4},\end{aligned}$$

such that

(i) $b_j \neq b_i \in \mathbb{C}\setminus\{0,1\}$,
(ii) $8l_{r+1} + 12l_{r+2} + 6l_{r+3} + 24(l_1+\cdots+l_r) \equiv 0 \mod (n)$ and
(iii) $\gcd(n, l_1, \ldots, l_{r+3}) = 1$.

If

$$\alpha(x,y) = (ix, i^{l_{r+3}/n}y),\ \beta(x,y) = (b(x), F(x)y),$$
$$F(x) = \frac{16^{l_{r+1}/n}\cdot(-64)^{l_{r+2}/n}\cdot(8i)^{l_{r+3}/n}\cdot 4096^{(l_1+\cdots+l_r))/n}}{(x+i)^{(8l_{r+1}+12l_{r+2}+6l_{r+3}+24(l_1+\cdots+l_r))/n}},$$

then

$$N = \langle \tau,\alpha,\beta:\ \tau^n=1,\ \alpha^4=\tau^{l_{r+3}},\ \beta^3=\tau^{l_{r+1}+l_{r+2}+l_{r+3}+l_1+\cdots+l_r},$$
$$(\alpha\beta)^2=\tau,\ \tau\alpha=\alpha\tau,\ \tau\beta=\beta\tau\rangle.$$

The signature of $\mathcal{X}/H$ is

$$\left(0; \frac{n}{n_{r+1}}, \overset{8}{\dots}, \frac{n}{n_{r+1}}, \frac{n}{n_{r+2}}, \overset{12}{\dots}, \frac{n}{n_{r+2}}, \frac{n}{n_{r+3}}, \overset{6}{\dots}, \frac{n}{n_{r+3}}, \frac{n}{n_1}, \overset{24}{\dots}, \frac{n}{n_1}, \dots, \frac{n}{n_r}, \overset{24}{\dots}, \frac{n}{n_r}\right),$$

the signature of $\mathcal{X}/N$ is

$$\left(0; \frac{3n}{n_{r+1}}, \frac{2n}{n_{r+2}}, \frac{4n}{n_{r+3}}, \frac{n}{n_1}, \frac{n}{n_2}, \dots, \frac{n}{n_r}\right),$$

and the genus of $\mathcal{X}$ is

$$g = 1 + 12(1+r)n - 4n_{r+1} - 6n_{r+2} - 3n_{r+3} - 12\textstyle\sum_{j=1}^{r} n_j. \tag{14}$$

Case $\overline{\mathbf{N}} \cong \mathbf{A_5}$: In this case,

$$A_5 := \left\langle a(x) = \omega_5 x, b(x) = \frac{(1-\omega_5^4)x + (\omega_5^4 - \omega_5)}{(\omega_5 - \omega_5^3)x + (\omega_5^2 - \omega_5^3)}\right\rangle$$

and $\mathcal{X}$ has the form

$$\mathcal{X}:\quad y^n = R_1(x)^{l_{r+1}} R_2(x)^{l_{r+2}} R_3(x)^{l_{r+3}} \textstyle\prod_{j=1}^{r} (R_1(x)^3 - 1728 b_j R_3(x)^5)^{l_j}, \tag{15}$$

where

$$\begin{aligned}
R_1(x) &= -x^{20} + 228x^{15} - 494x^{10} - 228x^5 - 1,\\
R_2(x) &= x^{30} + 522x^{25} - 10005x^{20} - 10005x^{10} - 522x^5 + 1,\\
R_3(x) &= x(x^{10} + 11x^5 - 1),\\
f(x) &= \tfrac{R_1(x)^3}{1728 R_3(x)^5},
\end{aligned}$$

such that

(i) $b_j \neq b_i \in \mathbb{C} \setminus \{0,1\}$,
(ii) $20l_{r+1} + 30l_{r+2} + 12l_{r+3} + 60(l_1 + \cdots + l_r) \equiv 0 \mod (n)$,
(iii) $\gcd(n, l_1, \dots, l_{r+3}) = 1$.

$$N = \langle \tau,\alpha,\beta:\ \alpha^5 = \tau^{l_{r+3}},\ \beta^3 = \tau^l,\ (\alpha\beta)^3 = \tau^t\rangle,$$

such that $\alpha(x,y) = (a(x), \omega_5^{l_{r+3}/n} y)$ and $\beta(x,y) = (b(x), F(x)y)$, where $F(x)$ is a rational map satisfying

$$F(b^2(x))\cdot F(b(x))\cdot F(x) = \omega_n^l,$$

for a suitable $l \in \{0, \dots, n-1\}$, and

$$F(x)^n = T_1^{l_{r+1}+3(l_1+\cdots+l_r)}(x) T_2^{l_{r+2}}(x) T_3^{l_{r+3}}(x),$$

where $T_j(x) = \frac{R_j(b(x))}{R_j(x)}$, for $j = 1, 2, 3$.

The signature of $\mathcal{X}/H$ is

$$\left(0; \frac{n}{n_{r+1}}, \overset{20}{\dots}, \frac{n}{n_{r+1}}, \frac{n}{n_{r+2}}, \overset{30}{\dots}, \frac{n}{n_{r+2}}, \frac{n}{n_{r+3}}, \overset{12}{\dots}, \frac{n}{n_{r+3}}, \frac{n}{n_1}, \overset{60}{\dots}, \frac{n}{n_1}, \dots, \frac{n}{n_r}, \overset{60}{\dots}, \frac{n}{n_r}\right),$$

and the signature of $\mathcal{X}/N$ is

$$\left(0; \frac{3n}{n_{r+1}}, \frac{2n}{n_{r+2}}, \frac{5n}{n_{r+3}}, \frac{n}{n_1}, \frac{n}{n_2}, \dots, \frac{n}{n_r}\right),$$

and the genus of $\mathcal{X}$ is

$$g = 1 + 30(r+1)n - 10n_{r+1} - 15n_{r+2} - 6n_{r+3} - 30\sum_{j=1}^{r} n_j. \tag{16}$$

□

4. Computing cyclic n-gonal curves

Consider the collection $\mathcal{F}_g$ of all the tuples $(n, s; n_1, \dots, n_s)$ satisfying the following Harvey's conditions:

(1) $n \geq 2$, $s \geq 3$, $2 \leq n_1 \leq n_2 \leq \cdots \leq n_s \leq n$;
(2) n_j is a divisor of n, for each $j = 1, \dots, s$;
(3) $\operatorname{lcm}(n_1, \dots, n_{j-1}, n_{j+1}, \dots, n_s) = n$, for every $j = 1, \dots, s$;
(4) if n is even, then $\#\{j \in \{1, \dots, s\} : n/n_j \text{ is odd}\}$ is even;
(5) $2(g-1) = n\left(s - 2 - \sum_{j=1}^{s} n_j^{-1}\right)$.

For each tuple $(n, s; n_1, \dots, n_s) \in \mathcal{F}_g$ we consider the collection $\mathcal{F}_g(n, s; n_1, \dots, n_s)$ of tuples $(l_1, \dots, l_s)$ so that

(1) $l_1, \dots, l_s \in \{1, \dots, n-1\}$;
(2) $l_1 + \cdots + l_s \equiv 0 \mod (n)$;
(3) $\gcd(n, l_j) = n/n_j$, for each $j = 1, \dots, s$.

Now, for each such tuple $(l_1, \dots, l_s) \in \mathcal{F}_g(n, s; n_1, \dots, n_s)$ we may consider the epimorphism

$$\theta : \Delta = \langle c_1, \dots, c_s : c_1^{n_1} = \cdots = c_s^{n_s} = c_1 \cdots c_s = 1 \rangle \to C_n = \langle \tau \rangle : c_j \mapsto \tau^{l_j}. \tag{17}$$

Our assumptions ensure that the kernel $\Gamma = \ker(\theta)$ is a torsion free normal co-compact Fuchsian subgroup of Δ with $\mathcal{X} = \mathbb{H}/\Gamma$ a closed Riemann surface of genus g admitting a cyclic group $H \cong C_n$ as a group of conformal automorphisms with quotient orbifold $\mathcal{X}/H = \mathbb{H}/\Delta$; a genus zero orbifold with exactly s cone points of respective orders $n_1, \dots, n_s$. The surface $\mathcal{X}$ corresponds to a cyclic n-gonal curve

$$C(n, s; l_1, \dots, l_s; p_1, \dots, p_s) : \quad y^n = \prod_{j=1}^{s} (x - p_j)^{l_j}, \tag{18}$$

for pairwise different values $p_1, \dots, p_s \in \mathbb{C}$, and H generated by $\tau(x, y) = (x, \omega_n y)$.

Different tuples

$$(l_1, \dots, l_s), (l'_1, \dots, l'_s) \in \mathcal{F}_g(n, s; n_1, \dots, n_s)$$

might provide isomorphic pairs $(\mathcal{X}, H)$ and $(\mathcal{X}', H')$ (i.e., there is an isomorphism between the Riemann surfaces conjugating the cyclic groups). In general this is a difficult problem to determine if different tuples define isomorphic pairs. But, in the non-exceptional fully generalized superelliptic situation (see **??**) the uniqueness of the superelliptic cyclic group of level n permits us to see that $(\mathcal{X}, H)$ and $(\mathcal{X}', H')$ are isomorphic pairs if and only if the corresponding curves

$$C(n, s; l_1, \dots, l_s; p_1, \dots, p_s) \quad \text{and} \quad C(n, s; l'_1, \dots, l'_s; p'_1, \dots, p'_s)$$

are isomorphic, this last being equivalent to the existence of Möbius transformation $t \in \mathrm{PSL}_2(\mathbb{C})$, a permutation $\eta \in S_s$ and an element

$$u \in \{1, \dots, n-1\}, \quad \text{with} \quad \gcd(u, n) = 1,$$

such that

(a) $l'_j \equiv ul_{\eta(j)} \mod (n)$, for $j = 1, \ldots, s$,

(b) $p'_{\eta(j)} = t(p_j)$, for $j = 1, \ldots, s$.

The above (together with Theorem 1) may be used to construct all the possible (generalized) superelliptic curves of lower genus in a similar fashion as done in [8] for the superelliptic case.

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