POLYNOMIALS, GALOIS GROUPS, AND MACHINE LEARNING

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Abstract. We use neural networks to investigate properties of irreducible polynomials $f \in \mathbb{Q}[x]$ and especially their Galois groups $\mathrm{Gal}\,(f)$. While for relatively small degree $f(x)$ methods of determining $\mathrm{Gal}\,(f)$ are known, our goal is to quickly train models which work for any degree polynomial with reasonable high degree of accuracy.

1. Introduction

Galois theory has a special place in mathematics, because it is about something fundamental as solving a polynomial equation in one variable. When most people know how to use the quadratic formula, fewer would be able to remember formulas for solving cubics and quartics. Things get even more interesting when the degree is ≥ 5, since such formulas do not exist for a generic polynomial, even though they do exist for special polynomials (i.e. polynomials such that their Galois group is solvable).

For the experts it is clear how to make the jump from a solvable Galois group of the polynomial to the formulas. The solvable group provides a solvable tower of subgroups which corresponds to a solvable tower of subfields of the splitting field of the polynomial. This solvable tower of subfields has cyclic extensions in every step and therefore corresponds to algebraic substitutions of the form $u = x^n$. This process is known as solving the polynomial by radicals. It is a rather complicated process when one tries to work out all the cases explicitly even for small degrees. One of the goals of this paper is to suggest a machine learning approach to find such formulas for higher degree polynomials. Then such formulas could be verified using Lean or some other format method.

Of course, the scope of Galois theory is much wider and deeper than figuring out formulas by radicals. Hence this use of machine learning in Galois theory can be used in a wide variety of methods and open questions. This paper is envisioned as a start of a large and long project of using data science in Galois theory.

[4, 13, 14, 16]

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2. Preliminaries

In this section we will go over some preliminary results on polynomials. Let k be a field. A degree $d \geq 1$ polynomial $f \in k[x]$ will be denoted by

$$f(x) = a_d x^d + a_{d-1} x^{d-1} y + \cdots + a_0 \tag{1}$$

Since we want to identify polynomials up to multiplication by a non-zero constant it is convenient sometimes to think of them in their projective form.

Let $k[x, y]$ be the polynomial ring in two variables and V_d denote the $(d+1)$-dimensional subspace of $k[x, y]$ consisting of homogeneous polynomials

$$f(x, y) = a_d x^d + a_{d-1} x^{d-1} y + \cdots + a_0 y^d \tag{2}$$

of degree d. Elements of V_d are called **binary forms** of degree d.

To every polynomial $f(x)$ we associate a binary form $f(x, y) = y^n f\left(\frac{x}{y}\right)$ as above, which is called the *homogenization of* $f(x)$. Conversely, every binary form $f(x, y)$ can be associated to a polynomial $f(x, 1)$, called the *dehomogenization of* $f(x, y)$.

2.1. Binary forms.

$\mathrm{GL}_2(k)$ acts as a natural group of automorphisms on $k[x, y]$. Denote by $f \to f^M$ this action. It is well known that $\mathrm{SL}_2(k)$ leaves a bilinear form (unique up to scalar multiples) on V_d invariant. If k is algebraically closed then $f(x, y)$ can be factored as

$$f(x, y) = (\beta_1 x - \alpha_1 y) \cdots (\beta_d x - \alpha_d y) = \prod_{1 \leq i \leq d} \det \left(\begin{pmatrix} x & \alpha_i \\ y & \beta_i \end{pmatrix} \right) \tag{3}$$

Points with homogeneous coordinates $(\alpha_i, \beta_i) \in \mathbb{P}^1$ are called the **projective roots** of f. For $M \in \mathrm{GL}_2(k)$ we have

$$f^M(x, y) = (\det M)^d \, (\beta_1' x - \alpha_1' y) \cdots (\beta_d' x - \alpha_d' y), \quad \text{where} \quad \begin{pmatrix} \alpha_i' \\ \beta_i' \end{pmatrix} = M^{-1} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}.$$

Consider $a_0, a_1, \ldots, a_d$ as transcendentals over k (coordinate functions on V_d). Then the coordinate ring of V_d can be identified with $k[a_0, \ldots, a_d]$. There is an action of $\mathrm{GL}_2(k)$ on $k[a_0, \ldots, a_d]$ via

$$\begin{aligned} &\mathrm{GL}_2(k) \times k[a_0, \ldots, a_d] \to k[a_0, \ldots, a_d] \\ &(M, F) \to F^M := F(f^M), \qquad \text{for all } \ f \in V_d. \end{aligned}$$

Thus for a polynomial $F \in k[a_0, \ldots, a_d]$ and $M \in \mathrm{GL}_2(k)$, define $F^M \in k[a_0, \ldots, a_d]$ as $F^M(f) := F(f^M)$, for all $f \in V_d$. Then $F^{MN} = (F^M)^N$. The homogeneous degree in $a_0, \ldots, a_d$ is called the **degree** of F, and the homogeneous degree in x, y is called the **order** of F. An **invariant** is usually referred to an $\mathrm{SL}_2(k)$-invariant on V_d. Hilbert's theorem says that the ring of invariants $\mathcal{R}_d$ is finitely generated. Thus, $\mathcal{R}_d$ is a finitely generated graded ring.

Let $\xi_0, \ldots, \xi_n$ be a minimal set of generators of $\mathcal{R}_d$ and $\deg \xi_i = q_i$. The set of degrees $(q_0, \ldots, q_n)$ is often called the **set of weights**.

Lemma 1. *Let $f, g \in V_d, M \in \mathrm{GL}_2(k), \lambda = (\det M)^{\frac{d}{2}}$. Then $f = g^M$ if and only if*

$$(\xi_0(f), \ldots \xi_i(f), \ldots, \xi_n(f)) = (\lambda^{q_0} \, \xi_0(g), \ldots, \lambda^{q_i} \, \xi_i(g), \ldots, \lambda^{q_n} \, \xi_n(g)).$$

If $k = \mathbb{Q}$ we can choose $\xi_0, \ldots, \xi_n \in \mathbb{Z}[a_0, \ldots, a_d]$ and primitive.

The theory of binary forms is quite extensive and well understood. However, the main goal of this paper is to construct a database of irreducible polynomials $f \in \mathbb{Q}[x]$ so we can study their Galois groups. Hence, we have to consider some other equivalences of polynomials.

2.2. **Equivalences of polynomials.** Notice that any polynomial $f \in \mathbb{Q}[x]$ can be written as $f = \lambda g(x)$ for some $g \in \mathbb{Z}[x]$. Since f and $g(x) = \lambda f(x)$ have the same Galois group, it is enough to consider only polynomials in $\mathbb{Z}[x]$.

Recall that $\mathrm{GL}_2(\mathbb{Z})$ is the subgroup of $\mathrm{GL}_2(\mathbb{Q})$ such that matrices have integer entries. Hence every matrix $M \in \mathrm{GL}_2(\mathbb{Z})$ has determinant $\det M = \pm 1$ and entries in $\mathbb{Z}$. .

Two polynomials $f, g \in \mathbb{Z}[x]$ of degree n are called **$\mathbb{Z}$-equivalent** if $f(x) = a^n\, g(ax+b)$ for some $a = \pm 1$ and $b \in \mathbb{Z}$.

Two degree n binary forms $f, g \in \mathbb{Z}[x, y]$ are called **$\mathrm{GL}_2(\mathbb{Z})$-equivariant** if $g(x, y) = \pm f(ax+by, cx+dy)$ for some $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z})$. Two degree n polynomials $f, g \in \mathbb{Z}[x]$ are called **$\mathrm{GL}_2(\mathbb{Z})$-equivalent** if their homogenizations are $\mathrm{GL}_2(\mathbb{Z})$-equivalent, in other words if

$$g(x) = \pm (cx+d)^n\, f\left(\frac{ax+d}{cs+d} \right), \qquad \text{for some} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}).$$

$f, g \in \mathbb{Q}[x]$ are called **$\mathbb{Q}$-equivalent** if $f(x) = g\left(\frac{ax+b}{cx+d}\right)$ for $a, b, c, d \in \mathbb{Q}$.

Lemma 2. *Let $f, g \in \mathbb{Z}[x]$. If f, g are $\mathbb{Z}$-equivalent, then they are $\mathrm{GL}_2(\mathbb{Z})$-equivalent and their homogenizations are $\mathrm{GL}_2(\mathbb{Q})$-equivalent.*

Hence the $\mathrm{GL}_2(\mathbb{Q})$ orbit, is partitioned into $\mathrm{GL}_2(\mathbb{Z})$-orbits and each $\mathrm{GL}_2(\mathbb{Z})$-orbit into $\mathbb{Z}$-orbits.

2.2.1. *Tschirnhaus-equivalent.* f and g (monic separable and irreducible of the same degree) are Tschirnhaus-equivalent iff they have the same splitting field E and moreover, if we let P and Q be the subgroups of $G := \mathrm{Gal}\ (E/k)$ fixing a root of f and g respectively, then P and Q are conjugate in G.

2.2.2. *Hermite equivalence.* Let $f(x) \in \mathbb{Z}[x]$ given as in Eq. (1) and $\alpha_1, \dots, \alpha_d \in \mathbb{C}$ its roots. Hence

$$f(x) = \sum_{i=0}^{d} a_i x^i = a_d \prod_{i=0}^{d} (x - \alpha_i)$$

To every root α_i we associate a linear form in new variables $x_1, \dots, x_d$ via

$$\alpha_i \to \alpha_i^{d-1} x_1 + \alpha_i^{d-2} x_2 + \cdots + \alpha_i x_{d-1} + x_d$$

Then we associate to f the d-ary form

$$f \longrightarrow a_d^{d-1} \prod_{i=1}^{d} \left(\alpha_i^{d-1} x_1 + \alpha_i^{d-2} x_2 + \cdots + \alpha_i x_{d-1} + x_d \right) =: [f]$$

The d-ary form $[f]$ is called the **Hermite form associated to** f. It is easy to show that the Hermite form is given by the resultant with respect to x of $f(x)$ and $g(x) = x_1 x^{d-1} + x_2 x^{d-2} + \cdots + x_{d-1} x + x_d$, namely

$$[f] = \mathrm{Res}(f, g, x)$$

Hence, $[f](x_1, \dots, x_d)$ is a d-ary form with integer coefficients. Moreover

$$\mathrm{cont}([f]) = (\mathrm{cont}(f))^{d-1}$$

Two polynomials $f, g \in \mathbb{Z}[x]$ of degree n are called **Hermite equivalent** if their corresponding Hermite forms are $\mathrm{GL}_n(\mathbb{Z})$-equivalent.

The discriminant of a decomposable d-ary form

$$F(x_1, \dots, x_d) = \prod_{i=1}^{d} (\alpha_{i,1} x_1 + \cdots + \alpha_{i,d} x_d)$$

is defined as

$$\Delta(F) = \left(\det\left(\alpha_{i,j}\right)_{i,j=1,\dots,d} \right)^2$$

Here are some properties of Hermitian forms. For proofs one can check [2].

Lemma 3. *The following are true:*

(i) *The discriminant of any polynomials is the same as the discriminant of its Hermite form. In other words,* $\Delta([f]) = \Delta_f$.
(ii) *Two polynomials which are Hermite equivalent have the same discriminants.*
(iii) *Let* $f, g \in \mathbb{Z}[x]$ *be* $\mathrm{GL}_2(\mathbb{Z})$*-equivalent polynomials. Then* f *and* g *are Hermite equivalent. Moreover, if* f *and* g *are monic and* $\mathbb{Z}$*-equivalent then they are Hermite equivalent.*
(iv) *(Hermite) There are finitely many Hermite equivalence classes of polynomials in* $\mathbb{Z}[x]$ *of a given degree and given discriminant* $\Delta \neq 0$.

2.2.3. *Julia equivalence.* Hermite defined the equivalence class of polynomials to develop a reduction theory for degree $d > 2$ polynomials. A reduction theory that was developed further by Julia; see [9] and [15, ?heights, ?min-gen-2, ?reduction-beshaj] for more recent treatments.

Let $f(x, y) \in \mathbb{Z}[x, y]$ be a degree n binary form

$$f(x, y) = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n$$

and suppose that $a_0 \neq 0$. Let the real roots of $f(x, y)$ be α_i, for $1 \leq i \leq r$ and the pair of complex roots β_j, $\bar{\beta}_j$ for $1 \leq j \leq s$, where $r + 2s = n$. The form can be factored as

$$f(x, 1) = \prod_{i=1}^{r} (x - \alpha_i) \cdot \prod_{i=1}^{s} (x - \beta_i)(x - \bar{\beta}_i). \tag{4}$$

The ordered pair (r, s) of numbers r and s is called the **signature** of the form f.

We associate to f the two quadratics $T_r(x, 1)$ and $S_s(x, 1)$ of degree r and s respectively given by the formulas

$$T_r(x, 1) = \sum_{i=1}^{r} t_i^2 (x - \alpha_i)^2, \quad \textit{and} \quad S_s(x, 1) = \sum_{j=1}^{s} 2u_j^2 (x - \beta_j)(x - \bar{\beta}_j), \tag{5}$$

where t_i, u_j are to be determined. For a binary form f of signature (r, s) the quadratic Q_f is defined as

$$\boxed{Q_f(x, 1) = T_r(x, 1) + S_s(x, 1).} \tag{6}$$

Let $\beta_i = a_i + b_i \cdot I$, for $i = 1, \ldots, s$.

The discriminant of Q_f is a degree 4 homogenous polynomial in $t_1, \ldots t_r$, $u_1, \ldots, u_s$. We pick values for $t_1, \ldots t_r, u_1, \ldots, u_s$ such that this discriminant is square free and minimal. Then we can use the reduction theory of quadratics (with square free, minimal discriminant) to determine the reduced form for Q_f. Define

$$\theta_T = \frac{a_0^2 \cdot \Delta_T}{t_1^2 \cdots t_r^2}, \qquad \theta_S = \frac{a_0^2 \cdot \Delta_S}{u_1^4 \cdots u_s^4} \tag{7}$$

Proposition 1. *Let* $f \in V_{n,\mathbb{Q}}$ *with signature* (r, s) *and equation as in Eq. (4). Then* Q_f *is a positive definite quadratic form with discriminant* $\mathfrak{D}_f$ *given by the formula*

$$\mathfrak{D}_f = \Delta(T_r) + \Delta(S_s) - 8 \sum_{i,j} t_i^2 u_j^2 \left((\alpha_i - a_j)^2 + b_j^2 \right). \tag{8}$$

From the above formula it can be seen that $\mathfrak{D}_f$ is expressed in terms of the root differences. Hence, $\mathfrak{D}_f$ is fixed by all the transpositions of the roots. However, it is not an invariant of the binary form. In order to get an invariant we need to fix it by all symmetries of the roots, hence by an element of order n. Indeed $\mathfrak{D}_f^n$ is an invariant of the binary form f as we will see later. We define the θ_0 of a binary form as follows

$$\theta_0(f) = \frac{a_0^2 \cdot |\mathfrak{D}_f|^{n/2}}{\prod_{i=1}^{r} t_i^2 \prod_{j=1}^{s} u_j^4}. \tag{9}$$

Notice that in order for f to be in somewhat "simpler" or "minimal" form we would like the discriminant $\mathfrak{D}_f$ to be minimal. Hence, we would like $\theta_0(f)$ to be minimal. Consider $\theta_0(t_1, \dots, t_r, u_1, \dots, u_s)$ as a multivariable function in the variables $t_1, \dots, t_r, u_1, \dots, u_s$. We would like to pick these variables such that Q_f is a reduced quadratic, hence $\mathfrak{D}_f$ is minimal. This is equivalent to $\theta_0(t_1, \dots, t_r, u_1, \dots, u_s)$ obtaining a minimal value.

Proposition 2. *The function $\theta_0 : \mathbb{R}^{r+s} \to \mathbb{R}$ obtains a minimum at a unique point $(\bar{t}_1, \dots, \bar{t}_r, \bar{u}_1, \dots, \bar{u}_s)$.*

Choosing $(\bar{t}_1, \dots, \bar{t}_r, \bar{u}_1, \dots, \bar{u}_s)$ that make θ_0 minimal gives a unique positive definite quadratic $Q_f(x, z)$. We call this unique quadratic $Q_f(x, z)$ for such a choice of $(\bar{t}_1, \dots, \bar{t}_r, \bar{u}_1, \dots, \bar{u}_s)$ the **Julia quadratic** of $f(x, z)$, denote it by $\mathcal{J}_f(x, z)$, and the quantity $\theta_f := \theta_0(\bar{t}_1, \dots, \bar{t}_r, \bar{u}_1, \dots, \bar{u}_s)$ the **Julia invariant**.

Lemma 4. *Consider $\mathrm{SL}_2(\mathbb{Q})$ acting on $V_{n,\mathbb{Q}}$. Then θ is an $\mathrm{SL}_2(\mathbb{Q})$- invariant and $\mathcal{J}$ is an $\mathrm{SL}_2(\mathbb{Q})$ covariant of order 2.*

Performing Julia reduction symbolically is very difficult, but a machine learning approach is used in [11] to perform Julia reduction to higher degree polynomials.

Hence, our database will have irreducible polynomials $f(x) \in \mathbb{Q}[x]$ (up to the above equivalence) which are represented as polynomials in $\mathbb{Z}[x]$. There are two main issues here:

i) identifying $\mathbb{Q}$-equivalence classes of polynomials,

ii) determining a method of listing and ordering such polynomials.

The first issue can be addressed via the classical invariant theory of binary forms, which motivates the material for the rest of this section. The second issue can be addressed via heights of polynomials which is the focus of next section.

2.3. **Proj $\mathcal{R}_d$ as a weighted projective space.** Let $\xi_0, \dots, \xi_n$ be the generators of $\mathcal{R}_d$ with degrees $q_0, \dots, q_n$ respectively. Since all $\xi_0, \dots, \xi_i, \dots, \xi_n$ are homogenous polynomials then $\mathcal{R}_d$ is a graded ring and Proj $\mathcal{R}_d$ as a weighted projective space.

Let $\mathbf{w} := (q_0, \dots, q_n) \in \mathbb{Z}^{n+1}$ be a fixed tuple of positive integers called **weights**. Consider the action of $k^\star = k \setminus \{0\}$ on $\mathbb{A}^{n+1}(k)$ as follows

$$\lambda \star (x_0, \dots, x_n) = (\lambda^{q_0} x_0, \dots, \lambda^{q_n} x_n)$$

for $\lambda \in k^*$. The quotient of this action is called a **weighted projective space** and denoted by $\mathbb{WP}^n_{(q_0,\dots,q_n)}(k)$. It is the projective variety *Proj* $(k[x_0, ..., x_n])$ associated to the graded ring $k[x_0, \dots, x_n]$ where the variable x_i has degree q_i for $i = 0, \dots, n$. We denote greatest common divisor of $q_0, \dots, q_n$ by $\gcd(q_0, \dots, q_n)$. The space $\mathbb{WP}^n_w$ is called **well-formed** if

$$\gcd(q_0, \dots, \hat{q}_i, \dots, q_n) = 1, \quad \text{for each} \ \ i = 0, \dots, n.$$

We l denote a point $\mathfrak{p} \in \mathbb{WP}^n_w(k)$ by $\mathfrak{p} = [x_0 : x_1 : \dots : x_n]$.

Let $\xi_0, \xi_1, \dots, \xi_n$ be the generators of the ring of invariants $\mathcal{R}_d$ of degree d binary forms. A k-isomorphism class of a binary form f is determined by the point

$$\xi(f) := [\xi_0(f), \xi_1(f), \dots, \xi_n(f)] \in \mathbb{WP}^n_{\mathbf{w}}(k).$$

Moreover, for any two forms f, and g we have that $f = g^M$ for some $M \in \mathrm{GL}_2(k)$ if and only if $\xi(f) = \lambda \star \xi(g)$, for $\lambda = (\det A)^{\frac{d}{2}}$.

2.4. **Generators of the ring of invariants.** Finding generators for the ring of invariants R_d is a classical problem of the XIX-century. Such generators are obtained in terms of transvections or root differences. Below we list the generating set of $\mathcal{R}_d$ for $d \leq 10$. From here on

$$f(x, y) = \sum_{i=0}^{d} \binom{d}{i} a_i x^i y^{d-i}$$

For given binary invariants $f, g \in V_d$ the r-th transvection of f and g is denoted by $(f, g)_r$.

While there is no method known to determine a generating set of invariants for any $\mathcal{R}_d$, we display a minimal generating set for all $3 \leq d \leq 10$. For the rest of this section $f(x,y)$ is given as in Eq. (2) and a minimal set of invariants is always picked as in lemma 1.

2.4.1. *Cubics.* A generating set for $\mathcal{R}_3$ is $\xi = \{\xi_0\}$, where

$$\xi_0 = ((f,f)_2,(f,f)_2)_2 = -54a_0^2a_3^2 + 36a_1a_3a_0a_2 - 8a_2^3a_0 - 8a_1^3a_3 + 2a_2^2a_1^2$$

2.4.2. *Quartics.* A generating set for $\mathcal{R}_4$ is $\xi = [\xi_0, \xi_1]$ with $\mathbf{w} = (2,3)$, where

$$\xi_0 = (f,f)_4 \quad \text{and} \quad \xi_1 = (f,(f,f)_2)_4$$

2.4.3. *Quintics.* A generating set for $\mathcal{R}_4$ is $\xi = [\xi_0, \xi_1, \xi_2]$ with $\mathbf{w} = (4,8,12)$, where

$$\xi_0 = (c_1,c_1)_2, \quad \xi_1 = (c_4,c_1)_2, \quad \xi_2 = (c_4,c_4)_2,$$

for $c_1 = (f,f)_4$, $c_2 = (f,f)_2$, $c_3 = (f,c_1)_2$, $c_4 = (c_3,c_3)_2$.

2.4.4. *Sextics.* The case of sextics was studied in detail by XIX-century mathematicians (Bolza, Clebsch, et al.) when char $k = 0$ and by Igusa for char $k > 0$. Let $c_1 = (f,f)_4$, $c_3 = (f,c_1)_4$, $c_4 = (c_1,c_1)_2$. A generating set for $\mathcal{R}_6$ is $\xi = [\xi_0, \xi_1, \xi_2, \xi_3]$ with weights $\mathbf{w} = (2,4,6,10)$, where

$$\xi_0 = (f,f)_6, \; \xi_1 = (c_1,c_1)_4, \; \xi_2 = (c_4,c_1)_4, \; \xi_3 = (c_4,c_3^2)_4$$

Usually the invariants of binary sextics are denoted by $[J_2, J_4, J_6, J_{10}]$ with J_{10} being the discriminant of the sextic, but that is not the case here.

2.4.5. *Septics.* A generating set of $\mathcal{R}_7$ is given by $\xi = [\xi_0, \xi_1, \xi_2, \xi_3, \xi_4]$ with weights $\mathbf{w} = (4,8,12,12,20)$. We define them as follows. Let

$$c_1 = (f,f)_6, \quad c_2 = (f,f)_4, \quad c_4 = (f,c_1)_2, \quad c_5 = (c_2,c_2)_4, \quad c_7 = (c_4,c_4)_4$$

$$\begin{aligned} &\xi_0 = (c_1,c_1)_2, && \xi_1 = (c_7,c_1)_2, && \xi_2 = ((c_5,c_5)_2,c_5)_4, \\ &\xi_3 = \left((c_4,c_4)_2,c_1^3\right)_6, && \xi_4 = \left([(c_2,c_5)_4]^2,(c_5,c_5)_2\right)_4. \end{aligned}$$

2.4.6. *Octavics.* A generating set of $\mathcal{R}_8$ is given by $\xi = [\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5]$ with weights $\mathbf{w} = (2,3,4,5,6,7)$. We define them as follows. Let

$$c_1 = (f,f)_6, \quad c_2 = (f,c_1)_4, \quad c_3 = (f,f)_4, \quad c_5 = (c_1,c_1)_2.$$

Then the invariants are:

$$\begin{aligned} &\xi_0 = (f,f)_8, && \xi_1 = (f,c_3)_8, && \xi_2 = (c_1,c_1)_4, && \xi_3 = (c_1,c_2)_4, \\ &\xi_4 = (c_5,c_1)_4, && \xi_5 = ((c_1,c_2)_2,c_1)_4. \end{aligned}$$

2.4.7. *Nonics.* A generating set of $\mathcal{R}_9$ is given by $\xi = [\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6]$ with weights $\mathbf{w} = (4,8,10,12,12,14,16)$. Let

$$\begin{aligned} &c_1 = (f,f)_8, \; c_2 = (f,f)_6, \; c_4 = (f,f)_2, \; c_5 = (f,c_1)_2, \; c_6 = (f,c_2)_6, \\ &c_7 = (c_2,c_2)_4, \; c_9 = (c_5,c_5)_4, \; c_{21} = (f,c_2)_2, \; c_{25} = (c_4,c_4)_{10}, \; c_{27} = (c_6^3,c_6)_3 \end{aligned}$$

$$\begin{aligned} &\xi_0 = (c_1,c_1)_2, && \xi_1 = (c_2,c_6^2)_6, && \xi_2 = (((c_{25},f)_6,c_{21})_5,c_2)_6, \\ &\xi_3 = ((c_7,c_7)_2,c_7)_4, && \xi_4 = (c_9,c_1^3)_6, && \xi_5 = ((c_2,c_{27})_3)_6, \\ &\xi_6 = \left((c_5,c_5)_2,c_1^5\right)_{10}. \end{aligned}$$

2.4.8. *Decimics.* A generating set of $\mathcal{R}_8$ is given by $\xi = [\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8]$ with weights $\mathbf{w} = (2, 4, 6, 6, 8, 9, 10, 14, 14)$. Let

$$\begin{array}{llll} c_1 = (f, f)_8, & c_2 = (f, f)_6, & c_5 = (f, c_1)_4, & c_6 = (f, c_2)_8, \\ c_7 = (c_2, c_2)_6, & c_8 = (c_5, c_5)_4, & c_9 = (c_2, c_7)_4, & c_{10} = (c_1, c_1)_2, \\ c_{16} = (c_5, c_5)_2, & c_{19} = (c_5, c_1)_1, & c_{25} = (c_7, c_7)_2 & \end{array}$$

$$\begin{array}{lll} \xi_0 = (f, f)_{10}, & \xi_1 = (c_1, c_1)_4, & \xi_2 = (c_5, c_5)_6, \\ \xi_3 = (c_6, c_6)_2, & \xi_4 = (c_1, c_8)_4, & \xi_5 = (c_{19}, c_1^2)_8), \\ \xi_6 = (c_{16}, c_1^2)_8, & \xi_7 = (c_{25}, c_9)_4, & \xi_8 = (c_{10}^2, c_{16})_8. \end{array}$$

2.5. **Root differences.** Invariants can also be expressed in terms of root differences. For example the discriminant is given by

$$\Delta(f) = \prod_{i \neq j} (\alpha_i - \alpha_j).$$

An excellent article on invariants including root differences is [12]. Multiplicities of the roots determine the stability of the binary forms via the Hilbert-Mumford criterion; see [5].

(i) If f has a root of multiplicity $r > \frac{d}{2}$ then $\xi(f) = (\xi_0, \dots, \xi_n) = (0, \dots, 0)$.
(ii) If d is even, then all binary forms with a root of multiplicity $\frac{d}{2}$ have the same invariants.

2.6. **Heights and moduli heights.** Let K be a number field, $\mathcal{O}_K$ its ring of integers, and M_K the set of absolute values of K.

2.7. **Heights of polynomials.** A polynomial with n variables is denoted by

$$f(x_1, \dots, x_n) = \sum_{i=(i_1, \dots, i_n) \in I} a_i x_1^{i_1} \cdots x_n^{i_n}$$

where all $a_i \in K$, $I \subset \mathbb{Z}^{\geq 0}$, and I is finite. We use lexicographic ordering to order the terms in a given polynomial, and $x_1 > x_2 > \dots > x_n$. The **(affine) multiplicative height of** f is defined as follows

$$H_K^{\mathbb{A}}(f) = \prod_{v \in M_K} \max \left\{ 1, |f|_v^{n_v} \right\}, \quad \text{where} \quad |f|_v := \max_j \left\{ |a_j|_v \right\}$$

is called the **Gauss norm** for any $v \in M_K$. The **(affine) logarithmic height of** f is defined to be

$$h_K^{\mathbb{A}}(f) = h_K([1, \dots, a_j, \dots]_{j \in I}).$$

The **(projective) multiplicative height** is

$$H_K(f) = \prod_{v \in M_K} |f|_v^{n_v} \tag{10}$$

where n_v is the completion of K_v; see [3] among other sources. The **(projective) absolute multiplicative height** is defined as

$$\begin{aligned} H : \mathbb{P}^n(\mathbb{Q}) &\to [1, \infty) \\ H(f) &= H_K(f)^{1/[K:\mathbb{Q}]}, \end{aligned}$$

It is a consequence of Northcott's theorem that for any $f(x, y) \in K[x, y]$. there are only finitely many polynomials $g(x, y) \in K[x, y]$ such that $H_K(g) \leq H_K(f)$.

Let $f(x_0, \dots, x_n)$ and $g(y_0, \dots, y_n)$ be polynomials in different variables. Then, the projective height has the following property

$$H(f \cdot g) = H(f) \cdot H(g)$$

Before considering the height of polynomials in the same variables, we will consider $|f \cdot g|_v$. The following lemma is true for the product of a finite number of polynomials.

Lemma 5 (Gauss's lemma)**.** *Let K be a number field and $f, g \in K[x_1, \ldots, x_n]$. If v is not Archimedean, then $|fg|_v = |f|_v |g|_v$.*

The proof can be found in [3, pg. 22].

An analogous Archimedean estimate is given by the following lemma. Gauss's lemma and the following are used to give an estimate of $H(f_1 f_2 \cdots f_r)$ in terms of $H(f_i)$ for $1 \leq i \leq r$ and $f_1, f_2, \ldots, f_r \in K[x_1, \ldots, x_n]$.

Lemma 6. *Let $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$, $f = f_1 \cdots f_r$, and $d_i = \deg(f, x_i)$. Then,*

$$\prod_{i=1}^{r} |f_i|_v \leq e^{(d_1 + \cdots + d_n)} |f|_v. \tag{11}$$

The proof of this can be found in [8, pg. 232] and uses the concept of Mahler measure which is defined as follows. Let $f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$. The **Mahler measure** is

$$M(f) := \exp\left(\int_{\mathbb{T}^n} \log \left| f(e^{i\theta_1}, \ldots, e^{i\theta_n}) \right| d\mu_1 \cdots d\mu_n \right)$$

where $\mathbb{T}$ is the unit circle $\{e^{i\theta} | 0 \leq \theta \leq 2\pi\}$ equipped with the standard measure $d\mu = \frac{1}{2\pi} d\theta$. Then $M(fg) = M(f)M(g)$; see [8, pg. 230] for proof.

Lemma 7. *Let K be a number field and $f_1, \ldots, f_r \in K[x_1, \ldots, x_n]$. Denote with $\deg f_j$ the total degree of f_j. Then the following are true*

(i) $H^{\mathbb{A}}(f_1 f_2 \cdots f_r) \leq N \cdot \prod_{j=1}^{r} H^{\mathbb{A}}(f_j) \leq r \cdot \max_{1 \leq j \leq r} \left\{ h(f_j) + (\deg f_j + m) \log 2 \right\}$.

(ii) $H^{\mathbb{A}}(f_1 + f_2 + \cdots + f_r) \leq r \cdot \prod_{j=1}^{r} H^{\mathbb{A}}(f_j)$.

(iii) *If $f_1, \ldots, f_r \in \mathcal{O}_K[x_1, \ldots, x_n]$, then*

$$H^{\mathbb{A}}(f_1 + f_2 + \cdots + f_r) \leq r \cdot \max_j \left\{ H^{\mathbb{A}}(f_j) \right\}^{[K:\mathbb{Q}]}.$$

The converse of part (i) is known as Gelfand's inequality.

Lemma 8 (Gelfand's inequality)**.** *Let $f_1, \ldots, f_r \in \overline{\mathbb{Q}}[x_1, \ldots, x_n]$, $d_i = \deg f_i$ such that $\deg(f_1 \cdots f_r, x_i) \leq d_i$ for each $1 \leq i \leq r$. Then*

$$\prod_{i=1}^{r} H(f_i) \leq e^{(d_i + \cdots + d_n)} \cdot H(f_1 \cdots f_r).$$

2.8. **Homogenous polynomials.** Next we focus on homogenous polynomials. For a fixed degree $d \geq 2$ and $f \in K[x_0, \ldots, x_n]$ we define

$$|c(d,n)|_v := \begin{cases} \binom{n+d}{n} & \text{if } v \text{ is Archimedean} \\ 1 & \text{if } v \text{ is non-Archimedean} \end{cases}$$

Lemma 9. *Let K be a number field, $f \in K[x_0, \ldots, x_n]$ a homogenous polynomial of degree d, and $\alpha = (\alpha_0, \ldots, \alpha_n) \in \overline{K}^{n+1}$. Then, the following hold:*

(1) $|f(\alpha)|_v \leq |c(d,n)|_v \cdot \max_j \left\{ |\alpha_j|_v \right\}^d \cdot |f|_v$, *where* $|c(d,n)|_v$ *is* $\binom{n+d}{d}$ *is v is non-Archimedean and 1 otherwise.*

(2) $H(f(\alpha)) \leq c_0 \cdot H(\alpha)^d \cdot H(f)$.

Lemma 9 can be used to determine the height of invariants of binary forms.

Corollary 1. *Let $f \in K[x,y]$ as in Eq. (2) and $\alpha = (\alpha_0, \alpha_1) \in \overline{K}^2$. Then,*

$$H(f(\alpha)) \leq \min\left\{d+1, 2^{d+1}\right\} \cdot H(\alpha)^d \cdot H(f).$$

2.9. **Minimal and moduli heights of forms.** Let $f(x,y)$ be a binary form and $Orb(f)$ its $GL_2(K)$-orbit in V_d. As a consequence of Northcott's theorem, there are only finitely many $f' \in Orb(f)$ such that $H(f') \leq H(f)$. Define the height of the binary form $f(x,y)$ as follows

$$\tilde{H}(f) := \min \left\{ H(f') | f' \in Orb(f),\, H(f') \leq H(f) \right\}$$

we want to consider the following problem. For every f let f' be the binary form such that $f' \in Orb(f)$ and $\tilde{H}(f) = H(f')$. Determine a matrix $M \in GL_2(K)$ such that $f' = f^M$.

Let $\mathcal{B}_d$ be the moduli space of degree d binary forms defined over an algebraically closed field k. Then $\mathcal{B}_d$ is a quasi-projective variety with dimension $d-3$. We denote the equivalence class of f by $\mathfrak{f} \in B_d$. The **moduli height** of $f(x,z)$ is defined as

$$\mathfrak{H}(f) = H(\mathfrak{f})$$

where $\mathfrak{f}$ is considered as a point in the projective space $\mathbb{P}^{d-3}$. A natural question would be to investigate if the minimal height $\tilde{H}(f)$ has any relation to the moduli height $\mathfrak{H}(f)$.

Let $\{I_{i,j}\}_{j=1}^{j=s}$ be a basis of $\mathcal{R}_d$. Here the subscript i denotes the degree of the homogenous polynomial $I_{i,j}$. The fixed field of invariants is the space $V_d^{GL_2(K)}$ and is generated by rational functions $t_1, \dots t_r$ where each of them is a ratio of polynomials in $I_{i,j}$ such that the combined degree of the numerator is the same as that of the denominator.

Theorem 2.1 ([?heights]). *Let f be a binary form. Then, For any $SL_2(k)$-invariant I_i of degree i we have that*

$$H\left(I_i(f)\right) \leq c \cdot H(f)^d \cdot H(I_i)$$

Moreover, $\mathfrak{H}(f) \leq c \cdot \tilde{H}(f)$, for some constant c.

For a given degree d the constant c of the theorem can be explicitly computed. For binary sextics i this constant is $c = 2^{28} \cdot 3^9 \cdot 5^5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 43$; see [?heights].

2.10. **Weighted moduli height.** For any point $\mathfrak{p} = [x_0 : \dots : x_n] \in \mathbb{P}^n_{\mathbf{w},k}$ we can assume, without loss of generality, that $\mathfrak{p} = [x_0 : \dots : x_n] \in \mathbb{P}^n_{\mathbf{w},k}(\mathcal{O}_k)$. Let $\mathbf{w} = (q_0, \dots, q_n)$ be a set of weights and $\mathbb{P}^n_{\mathbf{w},k}$ the weighted projective space over a number field k. Let $\mathfrak{p} \in \mathbb{P}^n_{\mathbf{w},k}$ a point such that $\mathfrak{p} = [x_0, \dots, x_n]$. We define the **weighted multiplicative height** of $\mathfrak{p}$ as

$$\mathfrak{H}_k(\mathfrak{p}) := \prod_{v \in M_k} \max \left\{ |x_0|_v^{\frac{n_v}{q_0}}, \dots, |x_n|_v^{\frac{n_v}{q_n}} \right\}. \tag{12}$$

The **absolute weighted height** of $\mathfrak{p} \in \mathbb{P}^n_{\mathbf{w},k}$ is the function $\mathfrak{H} : \mathbb{P}^n_{\mathbf{w},\overline{\mathbb{Q}}} \to [1, \infty)$,

$$\mathfrak{H}(\mathfrak{p}) = \mathfrak{H}_k(\mathfrak{p})^{1/[k:\mathbb{Q}]}, \tag{13}$$

where $\mathfrak{p} \in \mathbb{P}^n_{\mathbf{w},k}$, for any k which contains $\mathbb{Q}(\overline{wgcd}\,(\mathfrak{p}))$. The **absolute (logarithmic) weighted height** on $\mathbb{P}^n_{\mathbf{w},\overline{\mathbb{Q}}}$ is the function $\mathfrak{s} : \mathbb{P}^n_{\mathbf{w},\overline{\mathbb{Q}}} \to [0, \infty)$

$$\mathfrak{s}(\mathfrak{p}) = \log\, \mathfrak{H}_k(\mathfrak{p}) = \frac{1}{[k:\mathbb{Q}]}\, \mathfrak{H}_k(\mathfrak{p}).$$

where again $\mathfrak{p} \in \mathbb{P}^n_{\mathbf{w},k}$, for any k which contains $\mathbb{Q}(\overline{wgcd}\,(\mathfrak{p}))$.

Let $\mathbb{P}_{\mathbf{w},k}$ be a well-formed weighted projective space and $\mathbf{x} = [x_0 : \dots : x_n] \in \mathbb{P}_{\mathbf{w},k}(k)$. Assume $\mathbf{x}$ normalized (i.e. $\mathrm{wgcd}_k(\mathbf{x}) = 1$). Clearly $\mathrm{wgcd}(\mathbf{x}) | \gcd(x_0, \dots, x_n)$ and therefore $\mathrm{wgcd}(\mathbf{x}) \leq \gcd(x_0, \dots, x_n)$. Let $\mathbf{x}$ be absolutely normalized. Then $\gcd(x_0, \dots, x_n) = 1$. If $\mathbf{x} = [x_0 : \dots, x_n]$ is a normalized point then by definition of the height

$$\mathfrak{H}_k(\mathbf{x}) = \max_{i=0}^{n} \{ |x_i|^{\frac{1}{q_i}} \}$$

3. Galois groups of a polynomials

Let $\mathbb{F}$ be a perfect field. For simplicity we only consider the case when $\mathrm{char}\mathbb{F} = 0$. Let $f(x)$ be a degree $n = \deg f$ irreducible polynomial in $\mathbb{F}[x]$ which is factored as follows:

$$f(x) = (x - \alpha_1) \dots (x - \alpha_n) \tag{14}$$

in a splitting field E_f. Then, $E_f/\mathbb{F}$ is Galois because is a normal extension and separable. The group Gal $(E_f/\mathbb{F})$ is called **the Galois group** of $f(x)$ over $\mathbb{F}$ and denoted by Gal $_\mathbb{F}(f)$. The elements of Gal $_\mathbb{F}(f)$ permute roots of $f(x)$. Thus, the Galois group of polynomial has an isomorphic copy embedded in S_n, determined up to conjugacy by f. The main goal of this section is to determine Gal $_\mathbb{F}(f)$. Gal $_\mathbb{F}(f)$ can be viewed as a permutation group of the roots $\alpha_1, \dots, \alpha_n$. Thus it is a subgroup of S_n, determined up to conjugacy by f.

Proposition 3. *The following are true:*

(i) $\deg f \mid |G|$
(ii) *Let $G = Gal_\mathbb{F}(f)$ and $H = G \cap A_n$. Then $H = Gal\ (E_f/\mathbb{F}(\sqrt{\Delta_f}))$. In particular, G is contained in the alternating group A_n if and only if the discriminant Δ_f is a square in $\mathbb{F}$.*
(iii) *The irreducible factors of f in $\mathbb{F}[x]$ correspond to the orbits of G. In particular, G is a transitive subgroup of S_n if and only if f is irreducible.*

Proof. The first part is a basic property of the splitting field E_f. (ii) We have $\Delta_f = d_f^2$, where $d_f = \prod_{i>j}(\alpha_i - \alpha_j)$. For $g \in G$ we have $g(d_f) = \mathrm{sgn}(g) d_f$. Thus $H = G \cap A_n$ is the stabilizer of d_f in G. But this stabilizer equals Gal $(E_f/\mathbb{F}(d_f))$. Hence the claim.

(iii) G acts transitively on the roots of each irreducible factor of f. ☐

Lemma 10. *The following are true:*

(1) If $\sigma \in Gal\ (E_f/\mathbb{F})$ is a transposition then $\sigma(\Delta_f) = -\Delta_f$.
(2) If $\sigma \in Gal\ (E_f/\mathbb{F})$ is an even permutation then $\sigma(\Delta_f) = \Delta_f$.
(3) $Gal\ (E_f/\mathbb{F})$ is isomorphic to a subgroup of A_n if and only if $\Delta_f \in \mathbb{F}$.

When $n = 2$ then $f(x) = a_2x^2 + a_1x + a_0$. Thus, $\Delta_f = a_1^2 - 4a_0a_2$. Hence Gal $(f) \cong A_2 = \{1\}$ if and only if Δ_f is a square.

Lemma 11. *Let $f(x) \in \mathbb{F}[x]$ be an irreducible polynomial of degree $\deg f = n$. Then $Gal_\mathbb{F}(x)$ is an affine invariant of $f(x)$. In other words, $Gal\ (f) \cong Gal\ (g)$ for any $g(x) = f(ax + b)$, for $a, b \in \mathbb{F}$ and $a \neq 0$.*

Let $f(x, y) \in \mathbb{F}[x, y]$ be a binary form of degree $\deg f = n$. Let $g(x) = f(x, 1)$. Can Gal (g) be characterized in terms of invariants of the binary form $f(x, y)$? From section 2.4 we know that invariants of binary forms do not change under linear substitutions. Also from lemma 11 is invariant under such substitutions. Hence, we must be able to determine Gal (g) in terms of invariants of $f(x, y)$. For the rest of this section we will see how this can be done explicitly for cubics, quartics, and quintics.

3.1. Cubics.

Let $f(x)$ be an irreducible cubic polynomial in $\mathbb{F}[x]$. From **??** we know that $[E_f : \mathbb{F}] = 3$ or 6. Hence, the Galois group Gal $_\mathbb{F}(f)$ is a subgroup of S_3 with order 3 or 6. Thus, Gal $_\mathbb{F}(f) \cong A_3$ if and only if Δ_f is a square in $\mathbb{F}$, otherwise Gal $_\mathbb{F}(f) \cong S_3$.

Lemma 12. *Let $f(x) \in \mathbb{F}[x]$ be an irreducible cubic. Then $G = A_3$ if and only if $\xi_0(f) = \Delta_f$ is a square in $\mathbb{F}$. Moreover, the following hold:*

(i) $\Delta_f > 0$ *if and only if f has three distinct real roots.*
(ii) $\Delta_f < 0$ *iff f has one real root and two non-real complex conjugate roots.*

Since both A_3 and S_3 are solvable, we should be able to determine formulas to give the roots of $f(x)$ in terms of radicals. These formulas are known as Cardano's formulas and we will skip them here.

Remark 1. *What we notice from the cubics is that we can determine the Galois group simply by condition on invariants. We will see next if that can be done for higher degree polynomials.*

3.2. **Quartics.** Let $f(x) \in \mathbb{F}[x]$ be an irreducible polynomial of degree 4. Then $G := \mathrm{Gal}\,(f)$ is a transitive subgroup of S_4. Further $4 \mid |G|$, see Prop. 3. So the order of G is 4, 8, 12, or 24. It can be easily checked that transitive subgroups of S_4 of order 4, 8, 12, or 24 are isomorphic to one of the following groups

$$C_4,\ D_4,\ V_4,\ A_4,\ S_4.$$

Consider the normalized polynomial

$$f(x) = x^4 + ax^2 + bx + c = (x - \alpha_1) \dots (x - \alpha_4) \tag{15}$$

with $a, b, c \in \mathbb{F}$. Let $E_f = \mathbb{F}(\alpha_1, \dots, \alpha_4)$ be the splitting field of f over $\mathbb{F}$. Since f has no x^3-term, we have $\alpha_1 + \cdots + \alpha_4 = 0$. We assume $\Delta_f \neq 0$, so $\alpha_1, \dots, \alpha_4$ are distinct. Let $G = \mathrm{Gal}\,_{\mathbb{F}}(f)$, viewed as a subgroup of S_4 via permuting $\alpha_1, \dots, \alpha_4$.

There are 3 partitions of $\{1, \dots, 4\}$ into two pairs. S_4 permutes these 3 partitions, with kernel

$$V_4 = \{(12)(34), (13)(24), (14)(23), id\}. \tag{16}$$

Thus $S_4/V_4 \cong S_3$, the full symmetric group on these 3 partitions. Associate with these partitions the elements

$$\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4, \quad \beta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4, \quad \beta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3 \tag{17}$$

of E_f. If $\beta_1 = \beta_2$ then $\alpha_1(\alpha_2 - \alpha_3) = \alpha_4(\alpha_2 - \alpha_3)$, a contradiction. Similarly, $\beta_1, \beta_2, \beta_3$ are 3 distinct elements. Then G acts as a subgroup of S_4 on $\alpha_1, \dots, \alpha_4$, and as the corresponding subgroup of $S_3 \cong S_4/V_4$ on $\beta_1, \dots, \beta_3$. Thus the subgroup of G fixing all β_i is $G \cap V_4$. This proves the following:

$$\begin{array}{c} E_f := \mathbb{F}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ \Big| {\scriptstyle \bar{G} = G \cap V_4} \\ E := \mathbb{F}(\beta_1, \beta_2, \beta_3) \\ \Big| {\scriptstyle d} \\ \mathbb{F} \end{array}$$

Lemma 13. *The subgroup $G \cap V_4 \leq G$ corresponds to the subfield $\mathbb{F}(\beta_1, \beta_2, \beta_3)$, which is the splitting field over $\mathbb{F}$ of the cubic polynomial (*cubic resolvent*)*

$$g(x) = (x - \beta_1)(x - \beta_2)(x - \beta_3) = x^3 - ax^2 - 4cx + -b^2 + 4ac \tag{18}$$

The roots β_i of the cubic resolvent can be found by Cardano's formulas. The extension $\mathbb{F}(\alpha_1, \dots, \alpha_4)/k(\beta_1, \beta_2, \beta_3)$ has Galois group $\leq V_4$, hence is obtained by adjoining at most two square roots to $\mathbb{F}(\beta_1, \beta_2, \beta_3)$. Moreover, $\Delta(f, x) = \Delta(g, x)$.

In general, for an irreducible quartic

$$f(x) = x^4 + ax^3 + bx^2 + cx + d$$

we can first eliminate the coefficient of x^3 by the substituting x with $x - \frac{a}{4}$. In terms of the binary forms this corresponds to the transformation

$$(x, y) \to \left(x - \frac{a}{4}y, y\right)$$

and the new quartic is f^M for $M = \begin{bmatrix} 1 & -a/4 \\ 0 & 1 \end{bmatrix}$. Since $M \in \mathrm{SL}_2(\mathbb{Q})$ then $\det M = 1$ and the invariants of f^M are the same as those of f, namely ♠♠♠ Tony: [complete it]

$$\xi_0(f) =$$

$$\xi_1(f) =$$

Moreover $g(x)$ is

$$g(x) := x^3 - bx^2 + (ac - 4d)x - a^2d + 4bd - c^2. \tag{19}$$

The discriminant of $f(x)$ is the same as the discriminant of $g(x)$ and is given below:

$$\begin{aligned}\Delta_f = & -27a^4d^2 + 18a^3bcd - 4a^3c^3 - 4a^2b^3d + a^2b^2c^2 + 144a^2bd^2 - 6a^2c^2d - 80ab^2cd \\ & + 18abc^3 + 16b^4d - 4b^3c^2 - 192acd^2 - 128b^2d^2 + 144bc^2d - 27c^4 + 256d^3\end{aligned} \tag{20}$$

We denote by $d := [\mathbb{F}(\beta_1, \beta_2, \beta_3) : \mathbb{F}]$. Then we have the following:

Lemma 14. *The Galois group of $f(x)$ is one of the following:*

(i) $d = 1 \iff G \cong V_4$.
(ii) $d = 3 \iff G \cong A_4$.
(iii) $d = 6 \iff G \cong S_4$.
(iv) *If $d = 2$ then we have*
 a) $f(x)$ is irreducible over $F \iff G \cong D_4$
 b) $f(x)$ is reducible over $F \iff G \cong C_4$

3.2.1. *Solving quartics.* The element $(\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)$ is fixed by $G \cap V_4$, hence lies in $K(\beta_1, \beta_2, \beta_3)$. We find

$$-(\alpha_1 + \alpha_2)^2 = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) = \beta_2 + \beta_3 \tag{21}$$

By this and symmetry we get **Ferrari's formulas**

$$\begin{aligned}\alpha_1 + \alpha_2 &= \sqrt{-\beta_2 - \beta_3} \\ \alpha_1 + \alpha_3 &= \sqrt{-\beta_1 - \beta_3} \\ \alpha_1 + \alpha_4 &= \sqrt{-\beta_1 - \beta_2}\end{aligned} \tag{22}$$

or

$$\begin{aligned}\alpha_1 &= \frac{\sqrt{-\beta_1 - \beta_2} + \sqrt{-\beta_1 - \beta_3} + \sqrt{-\beta_2 - \beta_3}}{2} \\ \alpha_2 &= \frac{-\sqrt{-\beta_1 - \beta_2} - \sqrt{-\beta_1 - \beta_3} + \sqrt{-\beta_2 - \beta_3}}{2} \\ \alpha_3 &= \frac{-\sqrt{-\beta_1 - \beta_2} + \sqrt{-\beta_1 - \beta_3} - \sqrt{-\beta_2 - \beta_3}}{2} \\ \alpha_4 &= \frac{\sqrt{-\beta_1 - \beta_2} - \sqrt{-\beta_1 - \beta_3} - \sqrt{-\beta_2 - \beta_3}}{2}\end{aligned} \tag{23}$$

This completes the case for the quartics.

3.3. **Quintics.** Now we are ready to handle quintics which has such a special case in the history of Galois theory.

Lemma 15. *Let $f(x) \in \mathbb{F}[x]$ be an irreducible quintic. Then its Galois group is one of the following C_5, D_5, $F_5 = AGL(1, 5)$, A_5, S_5.*

Proof. G is transitive, hence its 5-Sylow subgroup is isomorphic to C_5 (generated by a 5-cycle). If C_5 is not normal, then G has at least 6 of 5-Sylow subgroups; then $|G| \geq 6 \cdot 5 = 30$, hence $[S_5 : G] \leq 4$ which implies $G = S_5, A_5$. If C_5 is normal in G then G is conjugate either C_5, D_5 (dihedral group of order 10) or $F_5 = AGL(1, 5)$, the full normalizer of C_5 in S_5, of order 20 (called also the Frobenius group of order 20). $\square$

Remark 2. *If the discriminant of the quintic is a square in $\mathbb{F}$ then Gal (f) is contained in A_5. Hence, it is C_5, D_5, or A_5.*

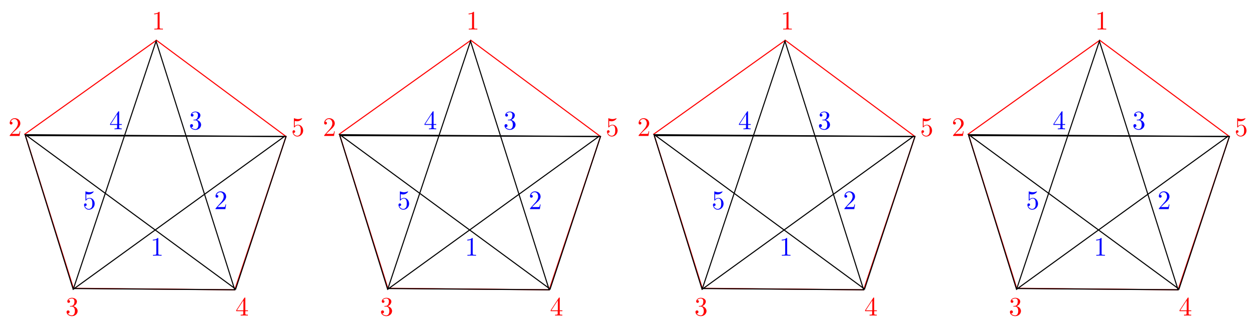
3.3.1. *Solvable quintics.* If $G = S_5, A_5$ then the equation $f(x) = 0$ is not solvable by radicals. We want to investigate here the case G is not isomorphic to S_5 or A_5. Let $f(x)$ be an irreducible quintic in $\mathbb{F}[x]$ given by

$$f(x) = x^5 + c_4x^4 + \cdots + c_0 = (x - \alpha_1)\cdots(x - \alpha_5) \tag{24}$$

Let $G = \text{Gal }(f)$, viewed as a (transitive) subgroup of S_5 via permuting the (distinct) roots $\alpha_1, \cdots, \alpha_5$. As before $E_f = \mathbb{F}(\alpha_1, \cdots, \alpha_5)$ denotes the splitting field.

A 5-cycle in $S_5 = \text{Sym}(\{1, \ldots, 5\})$ corresponds to an oriented pentagon with vertices $1, \ldots, 5$. A 5-cycle and its inverse correspond to a (non-oriented) pentagon, and the full C_5 corresponds to a pentagon together with its "opposite".

(a) (b) (c) (d)



Thus F_5, the normalizer of C_5 in S_5, is the subgroup permuting the pentagon and its opposite. D_5 is the subgroup of F_5 fixing the pentagon (symmetry group of the pentagon), and C_5 is the subgroup of rotations. For example, F_5 is generated by

$$F_5 = \langle \sigma, \tau \mid \sigma^5 = \tau^4 = (\sigma\tau)^4 = \sigma\sigma\tau\sigma^{-1}\tau^{-1} \rangle, \tag{25}$$

where $\sigma = (12345)$ and $\tau = (2453)$. Thus if $G \leq F_5$ then G fixes

$$\begin{aligned} \delta_1 &= (\alpha_1 - \alpha_2)^2(\alpha_2 - \alpha_3)^2(\alpha_3 - \alpha_4)^2(\alpha_4 - \alpha_5)^2(\alpha_5 - \alpha_1)^2 \\ &\quad - (\alpha_1 - \alpha_3)^2(\alpha_3 - \alpha_5)^2(\alpha_5 - \alpha_2)^2(\alpha_2 - \alpha_4)^2(\alpha_4 - \alpha_1)^2 \end{aligned} \tag{26}$$

where the first (resp., second) term corresponds to the edges of the pentagon (resp., its opposite). There are six 5-Sylow subgroups of S_5 given by

$$\begin{aligned}
H_1 &= \langle (1, 2, 3, 4, 5) \rangle = \{(), (1, 2, 3, 4, 5), (1, 3, 5, 2, 4), (1, 4, 2, 5, 3), (1, 5, 4, 3, 2)\} \\
H_2 &= \langle (1, 2, 3, 5, 4) \rangle = \{(), (1, 2, 3, 5, 4), (1, 3, 4, 2, 5), (1, 5, 2, 4, 3), (1, 4, 5, 3, 2)\} \\
H_3 &= \langle (1, 2, 4, 5, 3) \rangle = \{(), (1, 2, 4, 5, 3), (1, 4, 3, 2, 5), (1, 5, 2, 3, 4), (1, 3, 5, 4, 2)\} \\
H_4 &= \langle (1, 2, 4, 3, 5) \rangle = \{(), (1, 2, 4, 3, 5), (1, 4, 5, 2, 3), (1, 3, 2, 5, 4), (1, 5, 3, 4, 2)\} \\
H_5 &= \langle (1, 2, 5, 3, 4) \rangle = \{(), (1, 2, 5, 3, 4), (1, 5, 4, 2, 3), (1, 3, 2, 4, 5), (1, 4, 3, 5, 2)\} \\
H_6 &= \langle (1, 3, 4, 5, 2) \rangle = \{(), (1, 3, 4, 5, 2), (1, 4, 2, 3, 5), (1, 5, 2, 4, 3), (1, 5, 3, 2, 4)\}
\end{aligned}$$

To see the full invariance properties, we need to "projectivize" and use the invariants of binary forms; see section 2.4. Let $y = 1 = \beta_i$. The generalized version of the δ_1's is $\tilde{\delta}_1$, formed by replacing $\alpha_i - \alpha_j$ by $D_{ij} = \det \begin{bmatrix} \gamma_i & \beta_i \\ \gamma_j & \beta_j \end{bmatrix}$ in the formulas defining the δ_i's. In particular,

$$\tilde{\delta}_1 = D_{12}^2 D_{23}^2 D_{34}^2 D_{45}^2 D_{51}^2 - D_{13}^2 D_{35}^2 D_{52}^2 D_{24}^2 D_{41}^2 \tag{27}$$

Since S_5 has six 5-Sylow subgroups let $\delta_1, \ldots, \delta_6$ be the elements associated in this way to the six 5-Sylow's of S_5, i.e., to the six pentagon-opposite pentagon pairs on five given letters. We can write them all explicitly as

$$
\begin{aligned}
\tilde{\delta}_2 &= D_{12}^2 D_{23}^2 D_{35}^2 D_{54}^2 D_{41}^2 - D_{13}^2 D_{34}^2 D_{42}^2 D_{25}^2 D_{51}^2 \\
\tilde{\delta}_3 &= D_{12}^2 D_{24}^2 D_{45}^2 D_{53}^2 D_{31}^2 - D_{14}^2 D_{43}^2 D_{32}^2 D_{25}^2 D_{51}^2 \\
\tilde{\delta}_4 &= D_{12}^2 D_{24}^2 D_{43}^2 D_{35}^2 D_{51}^2 - D_{14}^2 D_{45}^2 D_{52}^2 D_{23}^2 D_{31}^2 \\
\tilde{\delta}_5 &= D_{12}^2 D_{25}^2 D_{53}^2 D_{34}^2 D_{41}^2 - D_{15}^2 D_{54}^2 D_{42}^2 D_{23}^2 D_{31}^2 \\
\tilde{\delta}_6 &= D_{13}^2 D_{34}^2 D_{45}^2 D_{52}^2 D_{21}^2 - D_{14}^2 D_{42}^2 D_{23}^2 D_{35}^2 D_{51}^2
\end{aligned} \tag{28}
$$

Lemma 16. *$\delta_i^\sigma = \delta_i$ dhe $\delta_i^\tau = \delta_i$ për $i = 1, \dots, 6$.*

Clearly, G permutes $\delta_1, \dots, \delta_6$. If G is conjugate to a subgroup of F_5, it fixes one of $\delta_1, \dots, \delta_6$; this fixed δ_i must then lie in $\mathbb{F}$.

Thus, a necessary condition for the (irreducible) polynomial $f(x)$ to be solvable by radicals is that one δ_i lies in $\mathbb{F}$, i.e., that the polynomial

$$
g(x) = (x - \delta_1) \cdots (x - \delta_6) \in \mathbb{F}[x] \tag{29}
$$

has a root in $\mathbb{F}$. It is also sufficient: If G fixes one δ_i then G is conjugate to a subgroup of F_5, provided that $\delta_1, \dots, \delta_6$ are all distinct. To check this is:

Exercise 1. *Show $\delta_1, \dots, \delta_6$ are mutually distinct (under the hypothesis $\Delta_f \neq 0$).*

The coefficients of $g(x)$ are symmetric functions in $\alpha_1, \dots, \alpha_5$, hence are polynomial expressions in $c_0, \dots, c_4$. The goal is to find these expressions explicitly. This gives an explicit criterion to check whether $f(x) = 0$ is solvable by radicals.

Lemma 17. *Let $s_r(x_1, \dots, x_6)$, $r = 1, \dots, 6$, be the elementary symmetric polynomials*

$$
s_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \dots x_{i_r}. \tag{30}
$$

Then $d_r := s_r(\tilde{\delta}_1, \dots, \tilde{\delta}_6)$ is a homogeneous polynomial expression in $b_0, \dots, b_5$ of degree $4r$. These polynomials are invariant under the action of $\mathrm{SL}_2(\mathbb{F})$ on binary quintics: For any $M \in \mathrm{SL}_2(\mathbb{F})$ the quintic f^M has the same associated d_r's.

Proof. For $\alpha_j := \gamma_j/\beta_j$ we have $\tilde{\delta}_i = (\beta_1 \cdots \beta_5)^4 \delta_i = b_5^4 \delta_i$. Thus $d_r = b_5^{4r} s_r(\delta_1, , \dots, \delta_6)$. But the $s_r(\delta_1, , \dots, \delta_6)$ are polynomial expressions in the $c_j = b_j/b_5$, for $j = 0, \dots, 4$. Thus d_r is a rational function in $b_0, \dots, b_5$, where the denominator is a power of b_5. Switching the roles of x and y yields that the denominator is also a power of b_0. Thus it is constant, i.e., d_r is a polynomial in $b_0, \dots, b_5$. If we replace each β_j by $c\beta_j$ for a scalar λ then each $\tilde{\delta}_i$ gets multiplied by λ^4, so d_r gets multiplied by λ^{4r}. Thus d_r is homogeneous of degree $4r$. The rest of the claim is clear. $\square$

(31)
$$
\begin{tikzcd}[column sep=small]
 & & & \mathbb{F}(\alpha_1, \dots, \alpha_5) \arrow[dlll, dash] \arrow[dll, dash] \arrow[dl, dash] \arrow[dr, dash] \arrow[drr, dash] \arrow[drrr, dash] & & & \\
\mathbb{F}(\delta_1) \arrow[drrr, dash] & \mathbb{F}(\delta_2) \arrow[drr, dash] & \mathbb{F}(\delta_3) \arrow[dr, dash] & & \mathbb{F}(\delta_4) \arrow[dl, dash] & \mathbb{F}(\delta_5) \arrow[dll, dash] & \mathbb{F}(\delta_6) \arrow[dlll, dash] \\
 & & & \mathbb{F}(\delta_1, \cdots, \delta_6) \arrow[d, dash] & & & \\
 & & & \mathbb{F}(s_1, \dots, s_6) \arrow[d, dash] & & & \\
 & & & \mathbb{F} & & &
\end{tikzcd}
$$

There are four basic invariants of quintics, denoted by J_4, J_8, J_{12}, J_{18}, of degrees 4,8,12 and 18, such that every $\mathrm{SL}(2, \mathbb{F})$-invariant polynomial in $b_0, \dots, b_5$ is a polynomial in J_4, J_8, J_{12}, J_{18}. To define J_4, J_8, J_{12}, we need auxiliary quantities

$$
\begin{aligned}
A &= \frac{1}{100}\left(20b_4 - 8b_1b_3 + 3b_2^2\right), \\
B &= \frac{1}{100}\left(100b_5 - 12b_1b_4 + 2b_2b_3\right), \\
C &= \frac{1}{100}\left(20b_1b_5 - 8b_2b_4 + 3b_3^2\right)
\end{aligned}
$$

and D, E, F, G defined by

$$
\begin{vmatrix}
10u + 2b_1v & 2b_1u + b_2v & b_2u + b_3v \\
2b_1u + b_2v & b_2u + b_3v & b_3u + 2b_4v \\
b_2u + b_3v & b_3u + 2b_4v & 2b_4u + 10b_5v
\end{vmatrix} = 10^3(Du^3 + Eu^2v + Fuv^2 + Gv^3)
$$

Then J_2, J_8, and J_{12} are given by

$$
\begin{aligned}
J_4 &= 5^3(B^2 - 4AC) \\
J_8 &= 2^5 \cdot 5^6 \left[2A(3EG - F^2) - B(9DG - EF) + 2C(3FD - E^2)\right] \\
J_{12} &= -2^{10} \cdot 5^9 \cdot 3^{-1}\left[4(3EG - F^2)(3FD - E^2) - (9DG - EF)^2\right]
\end{aligned} \tag{32}
$$

By using special quintics one gets linear equations for the coefficients expressing the d_r's in terms of J_4, J_8, J_{12}. The result is due to Berwick; see [10].

$$
\begin{aligned}
d_1 &= -10J_4 \\
d_2 &= 35J_4^2 + 10J_8 \\
d_3 &= -60J_4^3 - 30J_4J_8 - 10J_{12} \\
d_4 &= 55J_4^4 + 30J_4^2J_8 + 25J_8^2 + 50J_4J_{12} \\
d_5 &= -26J_4^5 - 10J_4^3J_8 - 44J_4J_8^2 - 59J_4^2J_{12} - 14J_8J_{12} \\
d_6 &= 5J_4^6 + 20J_4^2J_8^2 + 20J_4^3J_{12} + 20J_4J_8J_{12} + 25J_{12}^2
\end{aligned}
$$

Lemma 18. *Let $f(x)$ be a irreducible quintic over $\mathbb{F}$ and $d_1, \dots, d_6$ defined in terms of the coefficients of $f(x)$ as above. Then $f(x)$ is solvable by radicals if and only if $g(x) = x^6 + d_1 x^5 + \cdots d_5 x + d_6$ has a root in $\mathbb{F}$.*

Extending the method of invariants becomes harder for higher degree equations. For degree six equations see [1] and [7]. We are not aware of explicit computations for degree $d \geq 6$.

4. Reduction modulo p

The reduction method uses the fact that once a every polynomial with rational coefficients can be transformed into a monic polynomial with integer coefficients without changing the splitting field.

Let $f(x) \in \mathbb{Q}[x]$ be given by

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \tag{33}$$

Let d be the common denominator of all coefficients $a_0, \cdots, a_{n-1}$. Then $g(x) := df(\frac{x}{d})$ is a monic polynomial with integer coefficients. Clearly the splitting field of $f(x)$ is the same as the splitting field of $g(x)$. Thus, without loss of generality we can assume that $f(x)$ is a monic polynomial with integer coefficients.

Theorem 4.1. ***(Dedekind)*** *Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial such that $\deg f = n$, $Gal_{\mathbb{Q}}(f) = G$, and p a prime such that $p \nmid \Delta_f$. If $f_p := f(x) \mod p$ factors in $\mathbb{Z}_p[x]$ as a product of irreducible factors of degree $n_1, n_2, n_3, \cdots, n_k$, then G contains a permutation of type $(n_1)(n_2)\cdots(n_k)$*

The Dedekind theorem can be used to determine the Galois group in many cases since the *type* of permutation in S_n determines the conjugacy class in S_n. Consider for example polynomials of degree 5. The cycle types for all groups that occur as Galois groups of quintics are given below.

	(2)	$(2)^2$	(3)	(4)	(3)(2)	(5)
S_5	10	15	20	30	20	24
A_5		15	20			24
F_5		5		10		4
D_5		5				4
C_5						4

Table 1. Cycle types for Galois groups of quintics

In Table 2 we display the table for the type of elements in S_6. As it can be seen from the tables this method works well for degree 5 and 6. Unfortunately it does not work for degree $d > 6$.

	()	(2)	(2)(2)	(2)(2)(2)	(3)	(3)(2)	(3)(3)	(4)	(4)(2)	(5)	(6)	$\|G\|$
S_6	1	15	45	15	40	120	40	90	90	144	120	720
A_6	1	-	45	-	40	-	40	-	90	144	-	360
S_5	1	-	15	10	-	-	20	30	-	24	20	120
$(S_3 \times S_3) \rtimes C_2$	1	6	9	6	4	12	4	-	18	-	12	72
A_5	1	-	15	-	-	-	20	-	-	24	-	60
$C_2 \times S_4$	1	3	9	7	-	-	8	6	6	-	8	48
$(C_3 \times C_3) \rtimes C_4$	1	-	9	-	4	-	4	-	18	-	-	36
$S_3 \times S_3$	1	-	9	6	4	-	4	-	-	-	12	36
S_4	1	-	3	6	-	-	8	6	-	-	-	24
S_4	1	-	9	-	-	-	8	-	6	-	-	24
$C_2 \times A_4$	1	3	3	1	-	-	8	-	-	-	8	24
$C_3 \times S_3$	1	-	-	3	4	-	4	-	-	-	6	18
A_4	1	-	3	-	-	-	8	-	-	-	-	12
D_{12}	1	-	3	4	-	-	2	-	-	-	2	12
S_3	1	-	-	3	-	-	2	-	-	-	-	6
C_6	1	-	-	1	-	-	2	-	-	-	2	6

Table 2. Cycle types for Galois groups of sextics

5. Transitive groups

Here is the number of transitive subgroups for $n \leq 47$

n	# Subgroups	n	# Subgroups	n	# Subgroups	n	# Subgroups
5	5	6	16	7	7	8	50
9	34	10	45	11	8	12	301
13	9	14	63	15	104	16	1954
17	10	18	983	19	8	20	1117
21	164	22	59	23	7	24	25000
25	211	26	96	27	2392	28	1854
29	8	30	5712	31	12	33	162
34	115	35	407	36	121279	37	11
38	76	39	306	40	315842	41	10
42	9491	43	10	44	2113	45	10923

Table 3. Number of transitive subgroups of S_n for select values of n

Table 4. Transitive Subgroups of S_n for $n = 5, 6, 7, 11, 13, 17, 19$

n	Subgroups
5	$C(5) = 5, D(5) = 5:2, F(5) = 5:4, A_5, S_5$]
6	$C(6) = 6 = 3[x]^2, D_6(6) = [3]^2, D(6) = S(3)[x]^2, A_4(6) = [2^2]3, F_{18}(6) = [3^2]2 = 3 \wr 2,$ $2A_4(6) = [2^3]3 = 2 \wr 3, S_4(6d) = [2^2]S(3), S_4(6c) = \frac{1}{2}[2^3]S(3), F_{18}(6):2 = \left[\frac{1}{2}S(3)^2\right]2,$ $F_{36}(6) = \frac{1}{2}[S(3)^2]2, 2S_4(6) = [2^3]S(3) = 2 \wr S(3), L(6) = PSL(2,5) = A_5(6),$ $F_{36}(6):2 = [S(3)^2]2 = S(3) \wr 2, L(6):2 = PGL(2,5) = S_5(6), A_6, S_6$
7	$C(7) = 7, D(7) = 7:2, F_{21}(7) = 7:3, F_{42}(7) = 7:6, L(7) = L(3,2), A_7, S_7$
11	$C(11) = 11, D(11) = 11:2, F_{55}(11) = 11:5, F_{110}(11) = 11:10, L(11) = PSL(2,11)(11),$ $M(11), A_{11}, S_{11}$
13	$C(13) = 13, D(13) = 13:2, F_{39}(13) = 13:3, F_{52}(13) = 13:4, F_{78}(13) = 13:6,$ $F_{156}(13) = 13:12,$ $L(13) = PSL(3,3), A_{13}, S_{13}$
17	$C(17) = 17, D(17) = 17:2, F_{68}(17) = 17:4, F_{136}(17) = 17:8, F_{272}(17) = 17:16,$ $L(17) = PSL(2,16),\ L(17):2 = \mathrm{PZL}(2,16), L(17):4 = \mathrm{PYL}(2,16), A_{17}, S_{17}$
19	$C(19) = 19, D(19) = 19:2, F_{57}(19) = 19:3, F_{114}(19) = 19:6, F_{171}(19) = 19:9,$ $F_{342}(19) = 19:18, A_{19}, S_{19}$

6. Databases

6.1. Datasets of irreducible polynomials. In this section we want to create a database of irreducible polynomials $f \in \mathbb{Z}[x]$ of degree $\deg f = n$. Data will be stored in a Python dictionary. A polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ will be represented by its corresponding binary form $f(x, y) = \sum_{i=0}^{n} a_i x^i y^{n-i}$. Hence our points will be points in the projective space $\mathbb{P}^n_{\mathbb{Q}}$, i.e. points with integer coordinates

$$\mathfrak{p} = [a_n : \cdots : a_0] \in \mathbb{P}^n_{\mathbb{Q}},$$

such that $\gcd(a_0, \ldots, a_n)$. Since $f(x)$ is irreducible over $\mathbb{Q}$ and of degree $\deg f = n$, then $a_n \neq 0$ and $a_0 \neq 0$. Moreover, $\Delta_f \neq 0$.

6.2. Datasets with bounded height. Let us now trying to generate a dataset with a bounded height h as defined in Eq. (10). We will denote the set of such polynomials by $\mathcal{P}^h_n$. In other words

$$\mathcal{P}^h_n := \left\{ [a_n : \cdots : a_0] \in \mathbb{P}^n_{\mathbb{Q}} \;\mid\; a_0 a_n \neq 0, \Delta_f \neq 0, H_{\mathbb{Q}}([a_n : \cdots : a_0]) \leq h \right\}$$

where H_Q is defined as in Eq. (10). To ensure that the points in the database are not repeated we key the dictionary by the tuples $\left(1, \frac{a_{n-1}}{a_n}, \ldots, \frac{a_0}{a_n}\right)$. This is safe since $a_n \neq 0$. A dictionary in Python does not allow key duplicates, which ensures that there are no duplicates in our data. For given h, n the cardinality of $\mathcal{P}^h_n$ is bounded by

$$\#\mathcal{P}^h_n \leq 4h^2(2h+1)^{n-2}$$

The proof is a straightforward counting argument. There are more sophisticated methods to count algebraic points of bounded height on projective spaces; see for example [6] but we will work only over $\mathbb{Q}$ and our heights will be relatively small which does not allow for much redundant data.

For a degree $d \geq 3$ and height h one can use *Sagemath* and count such points as follows:

```
PP = ProjectiveSpace(d, QQ)
rational_points = PP.rational_points(h)
```

Below is the number of points for small d and h.

deg	h=1	h=2	h=3	h =4	h=5
3		136	668	1 940	4 936
4		694	4 823	18 528	569 912
5		3 616	34 860	174 120	639 476
6		18 602	249 498		

For every point $\mathfrak{p} = [a_n : \cdots : a_0]$ we will compute the following attributes

$$\left(1, \frac{a_{n-1}}{a_n}, \ldots, \frac{a_0}{a_n}\right) : [\mathfrak{p}, \xi_0, \ldots, \xi_n, \Delta_f, H(f), \mathfrak{H}_{\mathrm{k}}(\mathfrak{p}), \mathfrak{H}(\mathfrak{p}), \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_5, \mathcal{T}_7, \mathrm{Gal}_{\,\mathbb{Q}}(f), \mathrm{Relations},]$$

where

$\xi_0, \dots, \xi_n$	Invariants defined in *section* 2.4
Δ_f	Discriminant of $f(x)$
$H(f)$	Height of $f(x)$defined in *Eq.* (10)
$\mathfrak{H}_k(\mathfrak{p})$	Weighted moduli height as in *Eq.* (12)
$\mathfrak{H}(\mathfrak{p})$	Absolute weighted moduli height as in *Eq.* (13)
$\mathcal{T}_2$	Permutation type obtained by factorization modulo 2
$\mathcal{T}_3$	Permutation type obtained by factorization modulo 3
$\mathcal{T}_5$	Permutation type obtained by factorization modulo 5
$\mathcal{T}_7$	Permutation type obtained by factorization modulo 7
Gal $_{\mathbb{Q}}(f)$	Gap Identity of the Galois group of $f(x)$
Relations	Relations among invariants when possible determined by ....

After we complete this database we .....

TABLE 5. Irreducible degree 3 polynomials of height ≤ 5 and Galois group C_3

#	f	Δ	#	f	Δ	#	f	Δ
1	(1, 3, -4, 1)	7^2	15	(-1, -3, 0, 3)	3^4	29	(1, 2, -5, 1)	19^2
2	(-1, -4, -3, 1)	7^2	16	(1, -3, 0, 3)	3^4	30	(-1, -5, -2, 1)	19^2
3	(1, -1, -2, 1)	7^2	17	(5, 4, -5, 1)	13^2	31	(1, -5, 2, 1)	19^2
4	(1, -2, -1, 1)	7^2	18	(1, 1, -4, 1)	13^2	32	(-1, 2, 5, 1)	19^2
5	(-1, -2, 1, 1)	7^2	19	(5, -3, -2, 1)	13^2	33	(2, -1, -5, 2)	31^2
6	(-1, -1, 2, 1)	7^2	20	(-1, -4, -1, 1)	13^2	32	(2, -5, -1, 2)	31^2
7	(1, -4, 3, 1)	7^2	21	(1, -4, 1, 1)	13^2	35	(-2, -5, 1, 2)	31^2
8	(-1, 3, 4, 1)	7^2	22	(-5, -3, 2, 1)	13^2	36	(-2, -1, 5, 2)	31^2
9	(1, 0, -3, 1)	3^4	23	(-1, 1, 4, 1)	13^2	37	(3, -4, -5, 3)	61^2
10	(3, 0, -3, 1)	3^4	24	(-5, 4, 5, 1)	13^2	38	(3, -5, -4, 3)	61^2
11	(-1, -3, 0, 1)	3^4	25	(-1, -5, -4, 5)	13^2	39	(-3, -5, 4, 3)	61^2
12	(1, -3, 0, 1)	3^4	26	(1, -2, -3, 5)	13^2	40	(-3, -4, 5, 3)	61^2
13	(-3, 0, 3, 1)	3^4	27	(-1, -2, 3, 5)	13^2			
14	(-1, 0, 3, 1)	3^4	28	(1, -5, 4, 5)	13^2			

Remark 3. *From 4936 polynomials of degree three and height* ≤ 5 *only forty of them have Galois group of order 3. The discriminant* Δ_f *of those forty polynomials has values* $\Delta_f = 7^2, 3^4, 13^2, 19^2, 31^2$*, and* 61^2 *as shown in the Table.*

The discriminant of the whole list takes values $-26695 \leq \Delta_f \leq 5925$*.*

Compare this to the formula which bounds the invariants

Table 6. Polynomials of degree $d = 3$ and height $h \leq 5$ and the number of Galois groups in each case.l

$n = 3$				
h	$\#\mathbb{P}_h^3(\mathbb{Q})$	# irred	A_3	S_3
1	40	12		
2	272	136	4	
3	1120	668	12	
4	2928	1940	20	
5	6928	4936	40	

Table 7. Polynomials of degree $4 \leq d \leq 5$ and height $h \leq 5$ and the number of Galois groups in each case.l

$n = 4$							
h	$\#\mathbb{P}_h^4(\mathbb{Q})$	# irred	[4,1]	[4,2]	[8,3]	[12,3]	S_4
1	121	34	2	2	10	-	20
2	1 441	694	2	10	114	4	564
3	8 161	4 823	4	25	422	32	4 340
4	27 841	18 528	24	90	1 318	52	17 044
5	78 721	56 500	42	142	2812	108	53 396
$n = 5$							
h	$\#\mathbb{P}_h^5(\mathbb{Q})$	# irred	C_5	D_5	$F(5)$	A_5	S_5
1	364	104	-	-	-	-	104
2	7 448	3 616	0	12	8	12	3 584
3	58 096	34 860	0	100	28	76	34 656
4	257 544	174 120	4	192	104	180	173 640
5	877 240	639 476	8	388	268	460	638 352

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