IRREDUCIBLE SEXTICS, INVARIANTS, AND THEIR GALOIS GROUPS

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Abstract. This paper investigates the interplay between irreducible sextic polynomials, their invariants, and associated Galois groups. We employ a blend of classical algebraic methods and modern computational techniques, including machine learning, to analyze these structures. Our key contributions include the development of extensive datasets for sextics with bounded coefficients, the examination of weighted moduli points, and the application of clustering and classification algorithms to reveal patterns in Galois group distributions. We explore implications for Malle's conjecture, offering new statistical insights into the prevalence of specific Galois groups based on polynomial height. This study not only advances theoretical understanding but also suggests potential practical applications in computational algebra.

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1. Introduction

The complex relationship between polynomial invariants, their degree, and Galois groups has long been a central theme in algebra. Sextic polynomials, in particular, present intriguing challenges due to their higher degree and the variety of possible Galois groups they can exhibit. This paper builds on our previous work, where we explored the intersection of Galois theory and machine learning in the context of polynomial solvability and invariants [34]. Here, we extend this research by focusing specifically on irreducible sextics, which are polynomials of degree six without rational roots, aiming to uncover deeper patterns in their structural and algebraic properties.

The motivation for this study stems from both the theoretical allure of understanding sextics' symmetry through their invariants and the practical implications for computational mathematics. Sextic polynomials can exhibit numerous transitive Galois groups, from the full symmetric group S_6 to smaller, less symmetric groups, making them a fertile ground for study with implications in areas like cryptography and coding theory.

This paper is a natural combination of methods used in [33] and [34] We create datasets of irreducible sextic polynomials with bounded heights, providing a foundation for empirical study. This involves generating polynomials and organizing them based on their coefficients' magnitude and invariant properties.

We also examine weighted moduli points. Here, we discuss how weighted moduli points, derived from polynomial invariants, classify sextics into equivalence classes, simplifying the study of their symmetries.

Moreover, we leverage machine learning techniques. Both unsupervised and supervised learning methods are applied to explore the distribution of Galois groups. This includes clustering analyses to identify natural groupings based on polynomial characteristics and predictive models for Galois group identification.

Finally, we connect our findings to Malle's Conjecture. We test and interpret our results in the light of Malle's conjecture, which predicts the frequency of polynomials with specific Galois groups as a function of height, offering insights into the distribution patterns of these groups.

2. Preliminaries

This section outlines the foundational concepts and tools used throughout the paper, emphasizing aspects specific to sextic polynomials and their Galois groups. For a broader introduction to these topics, we refer readers to [34].

2.1. Irreducible Polynomials and Galois Groups.

Let $\mathbb{Q}$ be the field of rational numbers and $f(x)$ be a degree six irreducible polynomial in $\mathbb{Q}[x]$ which is factored as follows:

$$f(x) = (x - \alpha_1) \dots (x - \alpha_n) \tag{1}$$

in a splitting field E_f. Then, $E_f/\mathbb{Q}$ is Galois because is a normal extension and separable. The group Gal $(E_f/\mathbb{Q})$ is called **the Galois group** of $f(x)$ over $\mathbb{Q}$ and denoted by Gal $_{\mathbb{Q}}(f)$ or simply Gal (f). The elements of Gal (f) permute roots of $f(x)$. Thus, the Galois group of polynomial has an isomorphic copy embedded in S_n, determined up to conjugacy by f. The main goal of this section is to determine Gal (f).

2.2. Transitive Subgroups of S_6.

The symmetric group S_6, which consists of all permutations of six elements, plays a key role in the study of irreducible sextic polynomials. A subgroup G of S_6 is called transitive if its action on the six roots of a polynomial is such that any root can be mapped to any other. This property is directly related to the irreducibility of the polynomial over $\mathbb{Q}$, as it ensures that no proper subset of roots remains invariant under the action of G (see [34]).

There are exactly 16 transitive subgroups of S_6, including the full symmetric group S_6, the alternating group A_6, and smaller subgroups such as cyclic and dihedral groups. Each of these subgroups corresponds to a distinct structure for the splitting field of the polynomial, determining specific algebraic properties of its roots. These subgroups can be visualized in a lattice diagram, which organizes them based on containment. For instance, the maximal subgroups of S_6 highlight reductions in the symmetry of a sextic polynomial.

Table 1. Transitive Subgroups of S_6

Index	**Group Order**	**Group Description**
1	[6, 2]	$C(6) = 6 = 3[x]2$
2	[6, 1]	$D_6(6) = [3]2$
3	[12, 4]	$D(6) = S(3)[x]2$
4	[12, 3]	$A_4(6) = [2^2]3$
5	[18, 3]	$F_{18}(6) = [3^2]2 = 3 \wr 2$
6	[24, 13]	$2A_4(6) = [2^3]3 = 2 \wr 3$
7	[24, 12]	$S_4(6d) = [2^2]S(3)$
8	[24, 12]	$S_4(6c) = \frac{1}{2}[2^3]S(3)$
9	[36, 10]	$F_{18}(6) : 2 = [\frac{1}{2}S(3)^2]2$
10	[36, 9]	$F_{36}(6) = \frac{1}{2}[S(3)^2]2$
11	[48, 48]	$2S_4(6) = [2^3]S(3) = 2 \wr S(3)$
12	[60, 5]	$L(6) = PSL(2,5) = A_5(6)$
13	[72, 40]	$F_{36}(6) : 2 = [S(3)^2]2 = S(3) \wr 2$
14	[120, 34]	$L(6) : 2 = PGL(2,5) = S_5(6)$
15	[360, 118]	A_6
16	[720, 763]	S_6

The classification of these transitive subgroups and the construction of their lattice were achieved using GAP, a computational algebra system. GAP's tools enabled efficient verification of group properties and facilitated the generation of the table and diagram summarizing these relationships. Since the Galois group of an irreducible sextic polynomial must be one of these transitive subgroups, this classification is essential for understanding the symmetries and invariants of such polynomials. By associating each sextic polynomial with its corresponding Galois group, deeper insights into its algebraic structure can be gained.

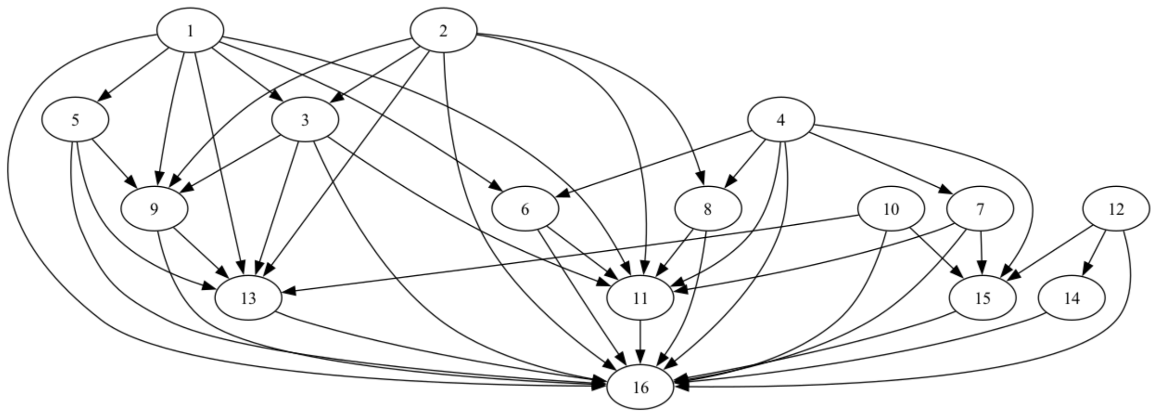

FIGURE 1. Lattice of transitive subgroups of S_6

2.3. **Reduction Modulo Primes:** Reduction modulo primes serves as a key tool for analyzing Galois groups. By examining the factorization patterns of polynomials modulo , we gain insight into the cycle types and subgroup structures of the associated Galois groups. This approach is a continuation of techniques outlined in [34]. Below we display the table for the type of elements in S_6.

	()	(2)	(2)(2)	(2)(2)(2)	(3)	(3)(2)	(3)(3)	(4)	(4)(2)	(5)	(6)	Order
S_6	1	15	45	15	40	120	40	90	90	144	120	720
A_6	1	-	45	-	40	-	40	-	90	144	-	360
S_5	1	-	15	10	-	-	20	30	-	24	20	120
$(S_3 \times S_3) \rtimes C_2$	1	6	9	6	4	12	4	-	18	-	12	72
A_5	1	-	15	-	-	-	20	-	-	24	-	60
$C_2 \times S_4$	1	3	9	7	-	-	8	6	6	-	8	48
$(C_3 \times C_3) \rtimes C_4$	1	-	9	-	4	-	4	-	18	-	-	36
$S_3 \times S_3$	1	-	9	6	4	-	4	-	-	-	12	36
S_4	1	-	3	6	-	-	8	6	-	-	-	24
S_4	1	-	9	-	-	-	8	-	6	-	-	24
$C_2 \times A_4$	1	3	3	1	-	-	8	-	-	-	8	24
$C_3 \times S_3$	1	-	-	3	4	-	4	-	-	-	6	18
A_4	1	-	3	-	-	-	8	-	-	-	-	12
D_{12}	1	-	3	4	-	-	2	-	-	-	2	12
S_3	1	-	-	3	-	-	2	-	-	-	-	6
C_6	1	-	-	1	-	-	2	-	-	-	2	6

TABLE 2. Cycle types for Galois groups of sextics

Notice that non-isomorphic transitive subgroups of S_6 have different cycle types, which makes determining the Galois groups of sextics relatively easier than higher degree polynomials. We label the above conjugacy classes above (not counting identity) as $C_1, \ldots, C_{10}$ and define the **signature** of G as the 10-tuple

$$\text{sig}(G) = [\alpha_1, \ldots, \alpha_{10}]$$

where $\alpha_i = 1$ if G has elements in C_i and $\alpha_i = 0$ otherwise. Hence,

TABLE 3. Transitive Subgroups of S_6

Index	Group Order	Group sig.
1	[6, 2]	[0,0,1,0,0,1,0,0,0,1]
2	[6, 1]	[0,0,1,0,0,1,0,0,0,0]
3	[12, 4]	[0,0,0,0,0,0,0,0,0,0]
4	[12, 3]	[0,0,0,0,0,0,0,0,0,0]
5	[18, 3]	[0,0,1,1,0,1,0,0,0,1]
6	[24, 13]	[0,0,0,0,0,0,0,0,0,0]
7	[24, 12]	[0,0,0,0,0,0,0,0,0,0]
8	[24, 12]	[0,0,0,0,0,0,0,0,0,0]
9	[36, 10]	[0,0,0,0,0,0,0,0,0,0]
10	[36, 9]	[0,0,0,0,0,0,0,0,0,0]
11	[48, 48]	[1,1,1,0,0,1,1,1,0,1]
12	[60, 5]	[0,1,0,0,0,1,0,0,1,0]
13	[72, 40]	[1,1,1,1,1,1,0,1,0,1]
14	[120, 34]	[0,1,1,0,0,1,1,0,1,1]
15	[360, 118]	[0,1,0,1,0,1,0,1,1,0]
16	[720, 763]	[1,1,1,1,1,1,1,1,1,1]

2.4. **Binary sextics and their invariants.** Following classical invariant theory, we represent sextic polynomials in their binary form , which facilitates the computation of invariants. These invariants, provide crucial insights into the structure and symmetries of the polynomial. To every polynomial $f(x)$ we associate a binary form

$$f(x, y) = y^n f\left(\frac{x}{y}\right)$$

as above, which is called the *homogenization of* $f(x)$. Conversely, every binary form $f(x, y)$ can be associated to a polynomial $f(x, 1)$, called the *dehomogenization of* $f(x, y)$.

Two degree n binary forms $f, g \in \mathbb{Z}[x, y]$ are called $\mathrm{GL}_2(\mathbb{Z})$**-equivariant** if $g(x, y) = \pm f(ax+by, cx+dy)$ for some $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z})$. Two degree n polynomials $f, g \in \mathbb{Z}[x]$ are called $\mathrm{GL}_2(\mathbb{Z})$**-equivalent** if their homogenizations are $\mathrm{GL}_2(\mathbb{Z})$-equivalent, in other words if

$$g(x) = \pm (cx + d)^n f\left(\frac{ax + d}{cs + d}\right), \quad \text{for some} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}).$$

$f, g \in \mathbb{Q}[x]$ are called $\mathbb{Q}$**-equivalent** if $f(x) = g\left(\frac{ax+b}{cx+d}\right)$ for $a, b, c, d \in \mathbb{Q}$.

We refer to [3] for notation and terminology. A binary sextic is given by

$$f(x, y) = a_6 x^6 + a_5 x^5 y + \cdots a_1 x y^5 + a_0 y^6$$

such that the discriminant of the sextic on the right is nonzero. Hence, the equivalence class of $f(x, y)$ is determined by the invariants of $f(x, y)$, which are commonly denoted by $J_2(f), J_4(f), J_6(f), J_{10}(f)$ and are homogenous polynomials of degree 2, 4, 6, and 10 respectively in the coefficients of $f(x, y)$. Moreover, the invariant $J_{10}(f)$ is the discriminant of the sextic and therefore $J_{10}(f) \neq 0$. Hence, the moduli space of binary sextics is isomorphic to $\mathbb{P}_{(2,4,6,10)} \setminus \{J_{10} \neq 0\}$.

The following invariants defined by Igusa in [15],

$$t_1 = \frac{J_2^5}{J_{10}}, \quad t_2 = \frac{J_4^5}{J_{10}^2}, \quad t_3 = \frac{J_6^5}{J_{10}^3} \tag{2}$$

are defined everywhere in the moduli space and are GL_2-invariants. Tow binary forms are $GL_2(\mathbb{Q})$-equivalent if and only if they have the same absolute invariants.

2.5. **Datasets and Bounded Heights:** Building on [34], we generate datasets of irreducible sextic polynomials with bounded heights. Heights are measured using the maximum absolute value of the polynomial's coefficients, ensuring a finite and manageable dataset. Points in the projective space represent these polynomials.

Our Python dictionary will be keyed on ordered triples (t_1, t_2, t_3), whcich are $\mathrm{GL}_2(\bar{k})$-invariants, hence every entry in the dictionary corresponds to the unique isomorphism class of genus 2 curves defined over $\bar{\mathbb{Q}}$.

	Entry	Value	Type	Description
0	(x, y, z)	(t_1, t_2, t_3)	float32	absolute invariants
1	$\mathfrak{p}$	$[J_2, J_4, J_6, J_{10}]$	int	normalized moduli point
2	$\bar{\mathfrak{p}}$		int	absolutely normalized point
3	wh	$\mathfrak{H}_\mathrm{k}(\mathfrak{p})$	float32	weighted height
4	awh	$\mathfrak{H}(\mathfrak{p})$	float32	absolute weighted height
5	gcd	$\gcd(\mathfrak{p})$	float32	gcd of $\mathfrak{p}$
6	label1	T/F	Boolean	True=fine, False=coarse
7	$[m, n]$	$Aut(\mathfrak{p})$	[int, int]	Gap Identity
8	label2	$\mathfrak{p} \in \mathcal{L}_3$	Boolean	
9	label3	$\mathfrak{p} \in \mathcal{L}_5$	Boolean	
10	label4	$\mathfrak{p} \in \mathcal{L}_7$	Boolean	

We will describe later how to normalize points $\mathfrak{p} = [a, b, c, d]$ in $\mathbb{P}_{(2,4,6,20)}$.

3. Databases

For a degree $d \geq 3$ and height h one can use *Sagemath* and count such points as follows:

```
PP = ProjectiveSpace(d, QQ)
rational_points = PP.rational_points(h)
```

For every point $\mathfrak{p} = [a_n : \cdots : a_0]$ we will compute the following attributes

$$\left(1, \frac{a_{n-1}}{a_n}, \dots, \frac{a_0}{a_n}\right) : [\mathfrak{p}, \xi_0, \dots, \xi_n, \Delta_f, H(f), \mathfrak{H}_k(\mathfrak{p}), \mathfrak{H}(\mathfrak{p}), \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_5, \mathcal{T}_7, \text{Gal}_{\mathbb{Q}}(f), \text{Relations},]$$

where

$\xi_0, \dots, \xi_n$	Invariants defined in **??**
Δ_f	Discriminant of $f(x)$
$H(f)$	Height of $f(x)$defined in **??**
$\mathfrak{H}_k(\mathfrak{p})$	Weighted moduli height as in **??**
$\mathfrak{H}(\mathfrak{p})$	Absolute weighted moduli height as in **??**
$\mathcal{T}_2$	Permutation type obtained by factorization modulo 2
$\mathcal{T}_3$	Permutation type obtained by factorization modulo 3
$\mathcal{T}_5$	Permutation type obtained by factorization modulo 5
$\mathcal{T}_7$	Permutation type obtained by factorization modulo 7
$\text{Gal}_{\mathbb{Q}}(f)$	Gap Identity of the Galois group of $f(x)$
Relations	Relations among invariants when possible determined by ....

Table 4. Polynomials of degree $d = 6$ and height $h \leq 5$ and the number of Galois groups in each case.l

$n = 6$															
h	$\#\mathbb{P}_h^6(\mathbb{Q})$	$f(x)$	C_6	G_2	G_3	S_3	G_5	G_6	G_7	G_8	G_9	G_{10}	G_{11}	A_6	G_{13}	G_{14}
1	1 093	292	4	-	-	-	-	-	-	-	2	20	-	-	8	18
2	37 969	18 602	4	2	12	-	12	-	8	10	72	106	8	4	348	316
3	409 585	249 498	4	6	80	14	36	12	44	60	192	396	26	84	1656	2156
4	2 351 329	1 613 884	8	12	272	30	128	168	180	120	30	1076	48	248	8640	6492
5	9 702 337	7 164 648	12	22	540	34	184	400	508	250	960	2262	98	672	18656	16770

Lemma 1. *Moreover, 20 sextics $f(x)$ with Gal $(f) \cong C_6$ and their* $\mathrm{SL}_2(\mathbb{Q})$*-invariants are:*

Notice that in the above table there are 2 or 4 polynomials corresponding to the same moduli point $\mathfrak{p}$. Counting them all shows that there are 4592 (1-times), 18183 (1-times), 4 (1-times), 2812 (1-times), 0 (1-times), 176 (1-times), 0 (1-times), 83 (1-times), 0 (1-times), 1 (1-times), 0 (1-times), 2 (1-times), for a total of 25 853 distinct moduli points.

There is however the possibility that two different points $\mathfrak{p}$ corresponds to the same moduli class because such points are up to equivalence in $\mathbb{WP}_{2,4,6,10}(\mathbb{Q})$. Next we will explain how to count such points in the weighted projective space.

Remark 1. *By using absolute invariants t_1, t_2, t_3 we can count the points in the moduli space and there are 25 853 such points.*

Lemma 2. *Polynomials with the same invariants (t_1, t_2, t_3) have the same Galois groups.*

#	$f(x)$	$\mathfrak{p}$
1	$x^6 - x^3 + 1$	$[-234, 1944, -129762, -19683]$
2	$x^6 + x^3 + 1$	
3	$x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$	$[-210, 1176, -76146, -16807]$
4	$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$	
5	$x^6 + 2x^5 + 5x^4 + 3x^2 + x + 1$	
6	$x^6 - 2x^5 + 5x^4 + 3x^2 - x + 1$	$[-400, 6076, -315952, -10955763]$
7	$x^6 + x^5 + 3x^4 + 5x^2 + 2x + 1$	
8	$x^6 - x^5 + 3x^4 + 5x^2 - 2x + 1$	
9	$x^6 + 2x^5 + 4x^4 + x^3 + 2x^2 - 3x + 1$	
10	$x^6 - 2x^5 + 4x^4 - x^3 + 2x^2 + 3x + 1$	$[-602, 14896, -2453136, -1075648]$
11	$x^6 + 3x^5 + 2x^4 - x^3 + 4x^2 - 2x + 1$	
12	$x^6 - 3x^5 + 2x^4 + x^3 + 4x^2 + 2x + 1$	
13	$x^6 + 6x^4 + 5x^2 + 1$	$[-720, 6468, -1435896, -153664]$
14	$x^6 + 5x^4 + 6x^2 + 1$	
15	$-x^6 - x^5 + 5x^4 + 4x^3 - 6x^2 - 3x + 1$	
16	$-x^6 + x^5 + 5x^4 - 4x^3 - 6x^2 + 3x + 1$	$[936, 10140, 2926404, 371293]$
17	$-x^6 - 3x^5 + 6x^4 + 4x^3 - 5x^2 - x + 1$	
18	$-x^6 + 3x^5 + 6x^4 - 4x^3 - 5x^2 + x + 1$	
19	$3x^6 - 6x^3 + 6x^2 - 3x + 1$	$[-504, 22356, -3327156, -26946027]$
20	$3x^6 + 6x^3 + 6x^2 + 3x + 1$	

TABLE 5. The only sextics with height $H \leq 6$ and Galois group C_6

Proof. complete it

$\square$

4. Fixing the Galois group

The main question we would like to ask now if whether our data satisfy any patterns for a fixed group $G = \text{Gal}(f)$.

4.1. Malle's conjecture.

Malle's conjecture deals with the distribution of number fields with a given Galois group and bounded discriminant. Specifically, it predicts an asymptotic formula for the number $N(G, X)$ of number field extensions L/K where:

- K is a number field,
- L is an extension of K with Galois group G,
- the norm of the discriminant of L/K is bounded by X.

Malle conjectured that for each fixed finite group G, there exist constants $c(G)$ and $a(G)$ such that:

$$N(G, X) \sim c(G) X^{a(G)} (\log X)^{b(G)}$$

where:

- $a(G) = \frac{1}{d(G)}$, where $d(G)$ is the smallest index of a subgroup of G that is not contained in any proper normal subgroup of G.
- $b(G)$ is related to the number of conjugacy classes of such subgroups.

This conjecture has been proven for specific groups like S_3, S_4, S_5 and for certain abelian groups, but counterexamples have been found for some groups, leading to modifications of the conjecture. The strong form of the conjecture specifies the constant $c(G)$ as well, but this has proven to be more challenging to verify universally.

Let us assume that we want to count irreducible polynomials of a given degree with bounded coefficients and specified Galois group. The goal is to estimate how many irreducible polynomials exist with specific Galois groups, as their heights (a measure of the size of their coefficients) grow.

Gal (f)	#
$F_{36}(6):2 = [S(3)^2]2 = S(3)wr2$	26 064
$2S_4(6) = [2^3]S(3) = 2wrS(3)$	20 236
$S_4(6d) = [2^2]S(3)$	2 608
$D(6) = S(3)[x]2$	1 185
$A6$	1 092
$L(6) = PSL(2,5) = A_5(6)$	706
$F_{18}(6):2 = [1/2.S(3)^2]2$	648
$L(6):2 = PGL(2,5) = S_5(6)$	534
$2A_4(6) = [2^3]3 = 2wr3$	394
$F_{18}(6) = [3^2]2 = 3wr2$	222
$S_4(6c) = 1/2[2^3]S(3)$	128
$F_{36}(6) = 1/2[S(3)^2]2$	58
$D_6(6) = [3]2$	43
$A_4(6) = [2^2]3$	34
$C(6) = 6 = 3[x]2$	20

Table 6. Galois groups and their frequencies for height $H \leq 6$.

Consider a finite group G, and let $N_G(h)$ denote the number of irreducible polynomials of a given degree n, with Galois group G and height $\leq h$. The analog of the Malle conjecture for polynomials predicts that:

$$N_g(h) \sim C_G \cdot h^k (\log h)^m ,$$

where C_G is a constant that depends on the group G, k and m are parameters related to the structure of the group G similar to how they appear in the original Malle conjecture.

Lemma 3. *There are exactly 53 972 irreducible sextics which have Galois group not isomorphic to S_6. The list of such groups and their frequencies are as follows:*

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